Asymptotic behaviour of Auslander-Reiten translations

Leon Lang

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Advisor: Prof. Dr. Jan Schröer

Second Advisor: Dr. Gustavo Jasso

Mathematical Institute

Mathematisch-Naturwissenschaftliche Fakultät der

Rheinischen Friedrich-Wilhelms-Universität Bonn

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Chapter 1

Introduction

Let k be an arbitrary (commutative) field and H be a finite-dimensional hereditary k-algebra. We denote by $\mathrm{mod}(H)$ the category of finite-dimensional left H-modules. All modules we consider are finite-dimensional.

1.1 The Auslander-Reiten translation and the Coxeter transformation

In our setting the Auslander-Reiten translations $\tau = D\,\mathrm{Tr} = D\,\mathrm{Ext}^1_H(-, H) : \mathrm{mod}(H) \to \mathrm{mod}(H)$ and $\tau^- = \mathrm{Tr}\,D = \mathrm{Ext}^1_{H^{\mathrm{op}}}(D-, H) : \mathrm{mod}(H) \to \mathrm{mod}(H)$ are defined, where $D = \mathrm{Hom}_k(-, k)$ is the k-dual functor and Tr is the transpose functor. τ is a left exact additive functor and τ^- is a right exact additive functor.

We denote by $\mathrm{mod}(H)_p$ and $\mathrm{mod}(H)_i$ the full subcategories of $\mathrm{mod}(H)$ consisting of all modules without projective and without injective direct summands, respectively. They are related by the Auslander-Reiten translations in the following way:

Proposition 1.1.1. *There is an exact equivalence of categories given by*

$$\mathrm{mod}(H)_i \underset{\tau}{\overset{\tau^-}{\rightleftarrows}} \mathrm{mod}(H)_p.$$

This restricts to an exact equivalence of the full subcategory of regular modules

$$\mathrm{reg}(H) \underset{\tau}{\overset{\tau^-}{\rightleftarrows}} \mathrm{reg}(H).$$

As a reference for all of this, see [ARS97, ch. IV], where the special features of hereditary algebras are discussed in Corollary 1.14 and Proposition 1.15.

The Auslander-Reiten translations are also important since they allow us to relate Hom-spaces to Ext-spaces by the Auslander-Reiten-formulas, see for example [ASS06, ch. IV.2, Corollary 2.14]:

Theorem 1.1.2 (Auslander-Reiten formula). *Let X and Y be H-modules. Then we have isomorphisms*

$$\operatorname{Ext}^1_H(X,Y) \cong D\operatorname{Hom}_H(Y,\tau X) \cong D\operatorname{Hom}_H(\tau^- Y, X)$$

which are functorial in X and Y.

Let $\{e_1,\dots,e_n\}$ be a complete set of primitive orthogonal idempotents in H. Then denote by $\{P(1),\dots,P(n)\}$, $\{I(1),\dots,I(n)\}$ and $\{S(1),\dots,S(n)\}$ sets of representatives of isomorphism classes of indecomposable projective modules, indecomposable injective modules and simple modules, respectively. Explicitly, we set $P(i) = He_i$, $I(i) = D(e_iH)$ and $S(i) = He_i/\operatorname{rad}(He_i) \cong D(e_iH/\operatorname{rad}(e_iH)) \cong \operatorname{top}(P(i)) \cong \operatorname{soc}(I(i))$, where $\operatorname{top}(P(i))$ denotes the *top* of $P(i)$ and $\operatorname{soc}(I(i))$ the *socle* of $I(i)$.

For an H-module M we denote by $\underline{\dim} M$ the *dimension vector* of M. It is a vector in $\mathbb{Z}^n$ which has as i'th entry the number of times the simple module $S(i)$ appears as a composition factor of M. Note that the order of the entries in a dimension vector, as well as the following definitions, depend on the choice of the order of the projectives, injectives, and simples, which we view as fixed from now on. Since H is of finite global dimension, all three sets $\{\underline{\dim} P(1),\dots,\underline{\dim} P(n)\}$, $\{\underline{\dim} I(1),\dots,\underline{\dim} I(n)\}$ and $\{\underline{\dim} S(1),\dots,\underline{\dim} S(n)\}$ form a basis of $\mathbb{Z}^n$, see [ARS97, ch. VIII.2]. The *Cartan matrix* of H is the invertible matrix

$$C_H := \left(\underline{\dim} P(1),\dots,\underline{\dim} P(n)\right) \in M^{n\times n}(\mathbb{Z}),$$

where we view $\underline{\dim} P(i)$ as a column vector. By [Rin84, ch. 2.4], the transpose of C_H is given by

$$C_H^t = \left(\underline{\dim} I(1),\dots,\underline{\dim} I(n)\right) \in M^{n\times n}(\mathbb{Z}),$$

and therefore the *Coxeter transformation* of H, defined by $\Phi_H := -C_H^t C_H^{-1}$, is given by

$$\Phi_H : \mathbb{Z}^n \to \mathbb{Z}^n,\ \underline{\dim} P(i) \mapsto -\underline{\dim} I(i),\ i \in \{1,\dots,n\}.$$

The importance of the Coxeter transformation comes from the following fact on hereditary finite-dimensional algebras, see [Rin84, ch. 2.4]:

Proposition 1.1.3. *Let M be an H-module.*

(i) If M lies in $\operatorname{mod}(H)_p$, then $\Phi_H \underline{\dim} M = \underline{\dim} \tau M$.

(ii) If M lies in $\operatorname{mod}(H)_i$, then $\Phi_H^{-1} \underline{\dim} M = \underline{\dim} \tau^- M$.

Therefore, in order to get information about the dimension vectors of τ-translated modules, it is reasonable to study the Coxeter transformation more closely.

1.2 The homological bilinear form

The *homological bilinear form* is given by

$$\langle -,- \rangle : \mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{Z},\ (x,y) \mapsto x^t C_H^{-t} y,$$

where we view x and y as column vectors (as we will always do when we deal with matrix multiplications) and where C_H^{-t} means the inverse of the transpose of C_H. It is called *homological* since it has the following homological interpretation when applied to dimension vectors (see [Rin84, p. 71] for a more general version for algebras that are not necessarily hereditary):

Proposition 1.2.1. *Let X and Y be H-modules. Then we have*

$$\left\langle \underline{\dim} X, \underline{\dim} Y \right\rangle = \dim_k \operatorname{Hom}_H(X,Y) - \dim_k \operatorname{Ext}^1_H(X,Y).$$

In particular, the right expression does only depend on the dimension vectors of X and Y.

The homological bilinear form has the following relation to the Coxeter transformation, which will in section 2.1 serve as a more abstract definition of what a Coxeter transformation is:

Proposition 1.2.2. *For all $x, y \in \mathbb{Z}^n$ we have*

$$\langle x, y \rangle = -\langle y, \Phi_H(x) \rangle = \langle \Phi_H(x), \Phi_H(y) \rangle$$

Proof. We just compute

$$-\langle y, \Phi_H(x)\rangle = -y^t C_H^{-t} \left(-C_H^t C_H^{-1}\right) x = y^t C_H^{-1} x = x^t C_H^{-t} y = \langle x, y \rangle,$$

where we used that 1×1-matrices are equal to its transpose. The second formula follows by applying the first formula twice. □

Let from now on $H = KQ$ be a finite-dimensional quiver algebra, where Q is a connected finite quiver without oriented cycles. Such algebras are hereditary, so everything we did so far applies. But in this case, the homological bilinear form can be computed directly from data of the quiver:

Proposition 1.2.3. *We identify $\{1, \dots, n\}$ with the vertex set Q_0. Then we have*

$$\langle x, y \rangle = \sum_{i \in Q_0} x_i y_i - \sum_{\alpha \in Q_1} x_{s(\alpha)} y_{t(\alpha)}.$$

In particular we get

$$\left\langle \underline{\dim} S(i), \underline{\dim} S(j) \right\rangle = \delta_{ij} - \#\{\alpha \in Q_1 \mid s(\alpha) = i,\ t(\alpha) = j\}$$

Proof. A proof using the homological description of $\langle -, - \rangle$ can be found in [GR97, ch. 7.2]. Since the dimension vectors of modules generate $\mathbb{Z}^n$, the general formula can be deduced from the formula on dimension vectors. □

The *Tits form* corresponding to the quiver Q is given by

$$q_Q : \mathbb{Z}^n \to \mathbb{Z},\ x \mapsto q_Q(x) = \langle x, x \rangle = \sum_{i \in Q_0} x_i^2 - \sum_{\alpha \in Q_1} x_{s(\alpha)} x_{t(\alpha)}.$$

The Tits form plays a role in classifying quivers in the following way:

Definition 1.2.4. Let Q be a connected quiver. Then

(i) Q is said to be of *Dynkin type* if q_Q is positive definite, i.e. $q_Q(x) > 0$ for all $0 \neq x \in \mathbb{Z}^n$.

(ii) Q is said to be of *Euclidean type* (or *extended Dynkin type* or *tame type*) if q_Q is positive semidefinite (i.e. $q_Q(x) \geq 0$ for all $x \in \mathbb{Z}^n$) but not positive definite.

(iii) Q is said to be of *wild type* if q_Q is indefinite, i.e. there are $y \neq 0 \neq x \in \mathbb{Z}^n$ such that $q_Q(x) < 0$ and $q_Q(y) > 0$. Since we always find $y \in \mathbb{Z}^n$ with $q_Q(y) > 0$, we do not need the second property.

The modules over $H = kQ$ are understood completely whenever Q is of Dynkin type or of Euclidean type, see [GR97, ch. 7, 10, 11]. But in the wild case a full classification is not possible, which is illustrated for example by the following Theorem (which we will neither prove, nor use), see [Ker96, Thm. 1.6]:

Theorem 1.2.5. *The path algebra $H = kQ$ of a connected wild quiver Q is strictly wild, i.e. for every finite-dimensional k-algebra B there is a full exact embedding of categories* $\mathrm{mod}(B) \to \mathrm{mod}(H)$.

Let Q and Q' two different wild connected quivers. Then $H = kQ$ and $H' = kQ'$ are according to the theorem both strictly wild algebras and therefore allow full exact embeddings $\mathrm{mod}(H) \to \mathrm{mod}(H') \to \mathrm{mod}(H)$. Therefore, a smaller version of $\mathrm{mod}(H)$ lies within $\mathrm{mod}(H)$ and in this sense the module category $\mathrm{mod}(H)$ is fractal. The theorem and this observation may sufficiently motivate the term *wild*. Nevertheless, $\mathrm{mod}(H)$ can still be studied relatively successful and in the following section we outline the known results on wild quivers that we want to present in this thesis.

1.3 Outline of known results and the line of action

Let $H = kQ$ be a finite-dimensional algebra, where Q is a connected wild quiver. Let $\mathrm{spec}(\Phi_H)$ be the spectrum of complex eigenvalues of the Coxeter transformation Φ_H (i.e. zeros of the characteristic polynomial), viewed as an isomorphism $\Phi_H : \mathbb{C}^n \to \mathbb{C}^n$ and let ρ_H be the spectral radius, i.e.

$$\rho_H = \max \{|\lambda| \, ; \lambda \in \mathrm{spec}(\Phi_H)\} .$$

We say that an eigenvalue is of *algebraic multiplicity one* or *simple* if its multiplicity as a root of the characteristic polynomial of Φ_H is one. We say a vector $v \in \mathbb{R}^n$ is *strictly positive* if every coordinate is positive, i.e. $v_i > 0$ for all $i \in \{1, \ldots, n\}$. In the wild setting, we will outline a proof of the following well-known result:

Theorem 1.3.1. *The spectral radius ρ_H has the following features:*

(i) $\rho_H > 1$ and ρ_H is itself an eigenvalue of Φ_H of multiplicity one.

(ii) If $\rho_H \neq \lambda \in \mathrm{spec}(\Phi_H)$, then $|\lambda| < \rho_H$.

(iii) There exist strictly positive vectors x^+ and x^- in $\mathbb{R}^n$ such that $\Phi_H x^+ = \rho_H x^+$ and $\Phi_H^{-1} x^- = \rho_H x^-$.

As a consequence, see Lemma 3.1.1, the growth behaviour of a vector $y \in \mathbb{C}^n$ after applying Φ_H many times will turn out to be completely determined by x^+, x^- and ρ_H. This will be a starting point for proving the following well-known theorem on the asymptotic behaviour of dimension vectors in the wild setting. It is sometimes also called exponential behaviour, since it is only asymptotic after correcting by the factor $\frac{1}{\rho_H^t}$ for $t \in \mathbb{N}$:

Theorem 1.3.2. *Assume X is a nonzero module without indecomposable preinjective direct summands and Y a nonzero module without indecomposable preprojective direct summands. Then we have the following:*

(i) There is a $\lambda_X^- > 0$ *such that* $\lim_{t\to\infty} \frac{1}{\rho_H^t} \underline{\dim} \tau^{-t} X = \lambda_X^- x^-$.

(ii) There is a $\lambda_Y^+ > 0$ *such that* $\lim_{t\to\infty} \frac{1}{\rho_H^t} \underline{\dim} \tau^{t} Y = \lambda_Y^+ x^+$.

(iii) $\lim_{t\to\infty} \frac{1}{\rho_H^t} \left\langle \underline{\dim} \tau^{-t} X, \underline{\dim} Y \right\rangle = \lim_{t\to\infty} \frac{1}{\rho_H^t} \left\langle \underline{\dim} X, \underline{\dim} \tau^{t} Y \right\rangle > 0$.

We will proceed in the following way: In Chapter 2 we give another definition of a Coxeter transformation in terms of generalized Cartan matrices and make a link to the Coxeter transformation of a wild path algebra. This will lead to an equivalent reformulation of Theorem 1.3.1 which we will prove following [dlP94] and [Rin94]. In the so-called *tree-case* (the case where Q does not have cycles) we also give a proof which was probably not stated in the literature elsewhere.

Then in Chapter 3 we go back to wild path algebras and deduce Theorem 1.3.2 and will gain further insight in the structure of the morphisms and regular components, mostly following [Ker96]. In the end of this chapter we will see applications of the developed theory following [KS02].

In Appendix A we collect some notation and terminology about quivers that we use throughout the text, in particular in the proof of the spectral properties of Coxeter transformations. In Appendix B we collect some facts about modules over general finite-dimensional hereditary algebras that we use throughout the text.

Of course, the theory provides also very interesting examples. We did not include many for space reasons and since the topics invite the reader to try out examples themselves. Nevertheless, one example will be examined in detail several times and will show very interesting properties, see Examples 2.1.24, 2.11.8, 3.3.3 and 3.5.9.

1.4 Note to the reader

The only purpose of Chapter 2 is to give a proof of Theorem 1.3.1. This proof is unfortunately very long. Since no insights from that chapter are used later (except the definition of preprojective and preinjective cones) and the only ingredient which will remain relevant in Chapter 3 is the *statement* of Theorem 1.3.1, a reader that is not particularly interested in long computations is advised to skip reading Chapter 2 completely. We also remark that we use three different versions of the classical Perron-Frobenius Theorem while proving Theorem 1.3.1, so there will in all cases remain something which has to be believed (unless the reader knows proofs of the Perron-Frobenius Theorems).

We also want to highlight the mathematics we assume the reader to be familiar with. The reader should have the knowledge of some lectures in the representation theory of (finite-dimensional) algebras. In particular, the reader should be somewhat familiar with the following concepts: Modules in general, path algebras, idempotents, socle, radical, semisimple modules, Jordan-Hölder Theorem, Krull-Remak-Schmidt Theorem, large and small modules, (pre-)projective and (pre-)injective and regular modules, the k-dual functor D, Homological algebra (mainly the interplay between Ext^1 and short exact sequences), Projective covers, Injective envelopes, Auslander-Reiten Theory (almost-split sequences – also called Auslander-Reiten sequences, irreducible morphisms, sink and source maps, Auslander-Reiten quiver, the transpose functor and the Auslander-Reiten translations, the interplay between all these notions). One of the best books about many of these topics is [ARS97].

Last but not least, we make some comments about the level of originality of this thesis. Most of it is just a survey with proofs of well-known results. Nevertheless, as mentioned, the second proof of the spectral properties of Coxeter transformations in the tree case, based on known eigenvector computations (which are partly incorrect, so we had to correct them), was probably not stated elsewhere in the literature. It can be found in section 2.5. Furthermore, the definition of a Coxeter transformation of a generalized Cartan matrix corresponding to a quiver, given in section 2.6, was probably not given elsewhere. This makes the arguments from Ringel given in [Rin94] more transparent. At one point we had to replace one of his arguments to fit the new framework. In this context we also gave a recursive formula of what a Coxeter transformation does in Lemma 2.8.6. Furthermore, in the proof that there is always a strictly positive eigenvector corresponding to the spectral radius of the Coxeter transformation, we carefully adapted the proof given in [dlPT90] – which only worked in the bipartite case – with help of our recursive formula and made it work in general. Furthermore the appendix sections B.4 and B.5 are maybe not stated in this generality before and served for understanding Kerners applications that we outline in section 3.6.

1.5 Acknowledgment

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Chapter 2

Spectral properties of Coxeter transformations

This chapter is devoted to the study of spectral properties of Coxeter transformations. Ultimately, we prove Theorem 1.3.1. First, we need to make a connection between Coxeter transformations of algebras and Coxeter transformations of generalized Cartan matrices, which we will introduce soon. Then we show that Coxeter transformations (with certain restrictions) always have an eigenvalue $\lambda > 1$. After we study the quiver corresponding to a Coxeter transformation (which is strongly related to the quiver of our algebra we started with) we give two different proofs of the spectral properties in the case that the quiver is a tree. The five sections afterwards are devoted to the study of the remaining case that there is a cycle in the quiver. We end this chapter by proving the existence of the strictly positive eigenvectors that are claimed by the theorem. We follow the convention that $e(i)$ always means the i'th standard basis vector in $\mathbb{R}^n$.

2.1 Linking different definitions of Coxeter transformations

First we introduce a general notion of what a Coxeter transformation ultimately is. Afterwards we introduce Coxeter transformations of generalized Cartan matrices. Finally we will see that Coxeter transformations of path algebras are the same as Coxeter transformations of *symmetric* generalized Cartan matrices. We follow [Lad08] in this section.

The Coxeter transformation of a bilinear form

Let $\langle -, - \rangle : \mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{Z}$ be a bilinear form.

Definition 2.1.1 (Coxeter transformation). A *Coxeter transformation* for $\langle -, - \rangle$ is a linear map $\Phi : \mathbb{Z}^n \to \mathbb{Z}^n$ such that for all $x, y \in \mathbb{Z}^n$ we have

$$\langle x, y \rangle = -\langle y, \Phi x \rangle .$$

Example 2.1.2. For a finite-dimensional hereditary algebra H, the Coxeter transformation Φ_H is a coxeter transformation for the homological bilinear form associated to H, see Proposition 1.2.2.

Definition 2.1.3 (Non-degenerate). The bilinear form $\langle -, - \rangle$ is called *non-degenerate* if the map $\mathbb{Z}^n \to (\mathbb{Z}^n)^*$, $y \mapsto (x \mapsto \langle x, y \rangle)$ is an isomorphism.

Proposition 2.1.4. *Assume that $\langle -, - \rangle$ is non-degenerate. Then there is a unique Coxeter transformation Φ for $\langle -, - \rangle$. Let $D = (D_{ij})$ be the matrix with $D_{ij} = \langle e(i), e(j) \rangle$. Then D is invertible and Φ is given by $\Phi = -D^{-1}D^t$.*

Proof. By definition of D we have $\langle x, y \rangle = x^t D y$ for all $x, y \in \mathbb{Z}^n$. We show that D is invertible: Assume $Dy = 0$. Then $\langle x, y \rangle = x^t D y = 0$ for all $x \in \mathbb{Z}^n$, and so $y = 0$ since $\langle -, - \rangle$ is non-degenerate. This shows that D is injective. Now let $y' \in \mathbb{Z}^n$. Then the map $\mathbb{Z}^n \to \mathbb{Z}$, $x \mapsto x^t y'$ lies in $(\mathbb{Z}^n)^*$, so since $\langle -, - \rangle$ is non-degenerate there is a $y \in \mathbb{Z}^n$ such that for all $x \in \mathbb{Z}^n$ we have $x^t D y = \langle x, y \rangle = x^t y'$. Since the standard scalar product is non-degenerate, we get $Dy = y'$ and thus D is surjective. It follows that D is invertible and therefore $\Phi := -D^{-1}D^t$ is well-defined. This is indeed a Coxeter transformation for $\langle -, - \rangle$ since we have for all $x, y \in \mathbb{Z}^n$

$$-\langle y, \Phi x \rangle = -y^t D(-D^{-1}D^t)x = y^t D^t x = x^t D y = \langle x, y \rangle,$$

where we used again that the transpose of a 1×1-matrix is the matrix itself. Finally, we show uniqueness: Assume that also $-\langle y, \Phi' x \rangle = \langle x, y \rangle$ for all $x, y \in \mathbb{Z}^n$. We get

$$-x^t (\Phi')^t D^t y = -y^t D \Phi' x = -\langle y, \Phi' x \rangle = \langle x, y \rangle = x^t D y.$$

It follows that $x^t \left[(\Phi')^t D^t + D \right] y = 0$ for all $x, y \in \mathbb{Z}^n$ and thus we see (by choosing the standard basis vectors for x and y) that $(\Phi')^t D^t + D = 0$. This implies $\Phi' = -D^{-1}D^t = \Phi$, proving uniqueness. □

For a finite-dimensional hereditary algebra H, the homological bilinear form is non-degenerate, since it is given by $\langle x, y \rangle = x^t C_H^{-t} y$ and since C_H^{-t} is invertible. Therefore, we could have introduced the Coxeter transformation Φ_H as the unique Coxeter transformation of the homological bilinear form. But by Proposition 2.1.4, this just means that $\Phi_H = -\left(C_H^{-t}\right)^{-1} \left(C_H^{-t}\right)^t = -C_H^t C_H^{-1}$, so we reconstruct the original definition.

The Coxeter transformation of a generalized Cartan matrix

Definition 2.1.5 (Generalized Cartan matrix). A *generalized Cartan matrix* $A \in M^{n \times n}(\mathbb{Z})$ is a matrix with integer entries such that the following three conditions hold for each $i \neq j$ in $\{1, \ldots, n\}$:

$$A_{ii} = 2, \quad A_{ij} \leq 0, \quad A_{ij} = 0 \Leftrightarrow A_{ji} = 0.$$

Note that generalized Cartan matrices need not be symmetric. Let from now on $A \in M^{n \times n}(\mathbb{Z})$ be a fixed generalized Cartan matrix. We always use the convention $\alpha_{ij} = -A_{ij}$. This convention serves as a tool for avoiding signs. The rules change to

$$\alpha_{ii} = -2, \quad \alpha_{ij} \geq 0, \quad \alpha_{ij} = 0 \Leftrightarrow \alpha_{ji} = 0.$$

Definition 2.1.6 (Reflection of A). Let $i \in \{1, \ldots, n\}$. Then the *i-th reflection* of A is defined on the standard basis vectors of $\mathbb{Z}^n$ by

$$R_i e(j) := e(j) - A_{ji} e(i) = e(j) + \alpha_{ji} e(i).$$

The following lemma explains why R_i is called a reflection:

Lemma 2.1.7. *For $i \in \{1, \ldots, n\}$, R_i is its own inverse.*

Proof. We have for all $j \in \{1, \ldots, n\}$:

$$\begin{aligned} R_i R_i e(j) &= R_i \left(e(j) + \alpha_{ji} e(i)\right) \\ &= e(j) + \alpha_{ji} e(i) + \alpha_{ji} \left(e(i) + \alpha_{ii} e(i)\right) \\ &= e(j) + 2 \cdot \alpha_{ji} e(i) - \alpha_{ji} \cdot 2 \cdot e(i) \\ &= e(j). \end{aligned}$$

Here we used the fact that generalized Cartan matrices have the property $\alpha_{ii} = -2$. □

Definition 2.1.8 (Coxeter transformation). Let $\pi : \{1, \ldots, n\} \to \{1, \ldots n\}$ be a permutation. Then we define the *Coxeter transformation for A* (with respect to π) to be the composition

$$C(A, \pi) : \mathbb{R}^n \to \mathbb{R}^n, \; x \mapsto R_{\pi(n)} \cdots R_{\pi(1)} x.$$

Lemma 2.1.9. *For any $m \in \{0, \ldots, n\}$ we have that $L := R_{\pi(m)} \cdots R_{\pi(1)}$ ($L :=$ id in case $m = 0$) is given by*

$$L(e(j)) = e(j) + \sum_{i=1}^{m} \left[\sum_{M} \alpha_{j\pi(k_1)} \cdot \alpha_{\pi(k_1)\pi(k_2)} \cdots \alpha_{\pi(k_{|M|})\pi(i)} \right] e(\pi(i)),$$

where $j \in \{1, \ldots, n\}$ and where the internal sum runs over all $M = \{k_1 < \cdots < k_{|M|}\} \subseteq \{1, \ldots, i-1\}$. In particular we can now compute $C(A, \pi)$ explicitly by setting $m = n$.

Proof. This is proven inductively: The case $m = 0$ is clear. Assume it is already proven for $m - 1$. Then it follows

$$\begin{aligned} L(e(j)) &= R_{\pi(m)} \left(e(j) + \sum_{i=1}^{m-1} \left[\sum_{M} \alpha_{j\pi(k_1)} \cdot \alpha_{\pi(k_1)\pi(k_2)} \cdots \alpha_{\pi(k_{|M|})\pi(i)} \right] e(\pi(i)) \right) \\ &= e(j) + \alpha_{j\pi(m)} e(\pi(m)) + \sum_{i=1}^{m-1} \left[\sum_{M} \alpha_{j\pi(k_1)} \cdot \alpha_{\pi(k_1)\pi(k_2)} \cdots \alpha_{\pi(k_{|M|})\pi(i)} \right] e(\pi(i)) \\ &\quad + \sum_{i=1}^{m-1} \left[\sum_{M} \alpha_{j\pi(k_1)} \cdot \alpha_{\pi(k_1)\pi(k_2)} \cdots \alpha_{\pi(k_{|M|})\pi(i)} \alpha_{\pi(i)\pi(m)} \right] e(\pi(m)) \\ &= e(j) + \sum_{i=1}^{m} \left[\sum_{M} \alpha_{j\pi(k_1)} \cdot \alpha_{\pi(k_1)\pi(k_2)} \cdots \alpha_{\pi(k_{|M|})\pi(i)} \right] e(\pi(i)). \end{aligned}$$

In the last step we observe that $\alpha_{j\pi(m)}$ is just the coefficient for the indices $i = m$ and $M = \emptyset$. □

Remark 2.1.10. Observe that we only have to sum over those coefficients i and M where the product $\alpha_{\pi(k_1)\pi(k_2)} \cdots \alpha_{\pi(k_{|M|})\pi(i)}$ (and also $\alpha_{j\pi(k_1)}$) is nonzero. We imagine that we find a path from $\pi(k_1)$ to $\pi(i)$ in this case: We only use increasing indices in $\{1, \ldots, n\}$ and only have an arrow $\pi(k) \to \pi(k')$ whenever $\alpha_{\pi(k)\pi(k')} \neq 0$. This will in fact be the definition of a quiver we will use extensively later.

Next we define the upper and lower triangular part of A and will then find another way of writing the Coxeter transformation of A. This will be the main step for linking it to the Coxeter transformations of algebras.

Definition 2.1.11 (Triangular parts of A). We define the *upper triangular part* A_+ and the *lower triangular* part A_- of A as follows:

$$(A_+)_{ij} := \begin{cases} A_{ji}, & i > j \\ 1, & i = j \\ 0, & \text{else} \end{cases} \quad \text{and} \quad (A_-)_{ij} := \begin{cases} A_{ij}, & i > j \\ 1, & i = j \\ 0, & \text{else} \end{cases}$$

Since $A_{ii} = 2$ we have $A = A_+^t + A_-$.

Lemma 2.1.12. *The matrices $A_+, A_- \in M^{n\times n}(\mathbb{Z})$ are invertible. Explicitly, we have*

$$A_{+/-}^{-1} = \sum_{k=0}^{n-1} (-1)^k \left(A_{+/-} - \mathrm{Id}\right)^k .$$

Proof. In any ring R, we clearly have the following: When $r \in R$ is nilpotent with $r^n = 0$, then $1 + r$ is invertible with inverse $\sum_{k=0}^{n-1}(-1)^k r^k$. The Lemma follows by observing that $(A_{+/-} - \mathrm{Id})^n = 0$. □

Theorem 2.1.13. *We have $-A_+^{-1}A_-^t = C(A, \mathrm{id})$.*

Proof. Using Lemma 2.1.12 we get

$$\begin{aligned} -A_+^{-1}A_-^t &= \mathrm{Id} - A_+^{-1}A_+ - A_+^{-1}A_-^t \\ &= \mathrm{Id} - A_+^{-1}(A_+^t + A_-)^t \\ &= \mathrm{Id} - A_+^{-1}A^t \\ &= \mathrm{Id} - \sum_{k=0}^{n-1}(-1)^k (A_+ - \mathrm{Id})^k A^t \end{aligned}$$

We claim that this coincides with $C(A, \mathrm{id}) = R_n \cdots R_1$. By evaluating the term on $e(j)$ for arbitrary $j \in \{1, \ldots, n\}$ and using Lemma 2.1.9 we see that this claim is equivalent to

$$\sum_{i=1}^{n} \left[\sum_M (-1)^{|M|+1} A_{jk_1} \cdots A_{k_{|M|}i} \right] e(i) = \sum_{k=0}^{n-1} (-1)^{k+1} (A_+ - \mathrm{Id})^k \begin{bmatrix} A_{j1} \\ \vdots \\ A_{jn} \end{bmatrix}.$$

We now show entrywise that those expressions are the same. The i-th entry of the left vector is

$$\sum_M (-1)^{|M|+1} A_{jk_1} \cdots A_{k_{|M|}i} = \sum_{s=0}^{i-1} (-1)^{s+1} \sum_{1 \le k_1 < \cdots < k_s < i} A_{jk_1} \cdots A_{k_s i}.$$

Since $(A_+ - \mathrm{Id})^s$ does not contribute to the i-th entry of the right side for $s \geq i$, it is enough that we prove for $s = 0, \ldots, i-1$ that

$$\sum_{1 \le k_1 < \cdots < k_s < i} A_{jk_1} \cdots A_{k_s i} = \left((A_+ - \mathrm{Id})^s \begin{bmatrix} A_{j1} \\ \vdots \\ A_{jn} \end{bmatrix} \right)_i$$

Let $B := A_+ - \mathrm{Id}$. Then

$$(B^s)_{ij} = \sum_{k_2,\dots,k_s} B_{ik_s} B_{k_s k_{s-1}} \cdots B_{k_2 j} = \sum_{j<k_2<\dots<k_s<i} A_{jk_2} \cdots A_{k_s i}$$

and thus

$$\begin{aligned}
\left((A_+ - \mathrm{Id})^s \begin{bmatrix} A_{j1} \\ \vdots \\ A_{jn} \end{bmatrix} \right)_i &= \left(B^s \begin{bmatrix} A_{j1} \\ \vdots \\ A_{jn} \end{bmatrix} \right)_i \\
&= \sum_{k_1=1}^{n} A_{jk_1} (B^s)_{ik_1} \\
&= \sum_{k_1=1}^{n} \sum_{k_1<k_2<\dots<k_s<i} A_{jk_1} \cdot A_{k_1 k_2} \cdots A_{k_s i} \\
&= \sum_{1\le k_1<\dots<k_s<i} A_{jk_1} \cdots A_{k_s i},
\end{aligned}$$

finishing the proof. □

Example 2.1.14. Look at the generalized Cartan matrix

$$A = \begin{pmatrix} 2 & 0 & -1 \\ 0 & 2 & -5 \\ -3 & -2 & 2 \end{pmatrix}.$$

We want to determine the Coxeter transformation $C(A, \mathrm{id})$. The reflections are given by

$$R_1 = \begin{pmatrix} -1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad R_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 2 \\ 0 & 0 & 1 \end{pmatrix}, \quad R_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 5 & -1 \end{pmatrix}.$$

Therefore, the Coxeter transformation is given by

$$C(A, \mathrm{id}) = R_3 R_2 R_1 = \begin{pmatrix} -1 & 0 & 3 \\ 0 & -1 & 2 \\ -1 & -5 & 12 \end{pmatrix}.$$

By Theorem 2.1.13 We can also compute the Coxeter transformation directly from the upper and lower triangular part. Let us test this: We have

$$-A_+^{-1} A_-^t = -\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 5 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & -3 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 3 \\ 0 & -1 & 2 \\ -1 & -5 & 12 \end{pmatrix} = C(A, \mathrm{id}),$$

which is what we expected.

Proof of equivalence between the definitions

Proposition 2.1.15. *Let D be a lower unitriangular matrix, i.e. $D_{ij} = 1$ for $i = j$ and $D_{ij} = 0$ for $i < j$. Assume further $D_{ij} \leq 0$ for all $i > j$. To D there corresponds a non-degenerate bilinear form $\langle -, - \rangle : \mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{Z}$. Let Φ be the Coxeter transformation of that bilinear form. Then $\Phi = C(D + D^t, \mathrm{id})$, i.e. it is the Coxeter transformation of the symmetric generalized Cartan matrix $D + D^t$.*

Proof. Again, since $D - \mathrm{Id}$ is nilpotent we get that D is invertible, see the proof of Lemma 2.1.12. Thus the Coxeter transformation Φ is defined according to Proposition 2.1.4. Let $A := D + D^t$. Then A is clearly a symmetric generalized Cartan matrix and thus by Theorem 2.1.13 and Proposition 2.1.4 we get

$$\Phi = -D^{-1}D^t = -A_+^{-1}A_-^t = C(A, \mathrm{id}) = C\left(D + D^t, \mathrm{id}\right),$$

which finishes the proof. □

Theorem 2.1.16. *Let $H = kQ$ be a finite-dimensional path algebra of a connected quiver Q. Then the Coxeter transformation Φ_H is up to conjugation by a permutation matrix of the form $C(A, \mathrm{id})$ for some symmetric generalized Cartan matrix A. In particular, the spectrum of eigenvalues of Φ_H and the spectrum of eigenvalues of $C(A, \mathrm{id})$ coincide.*

Proof. Let $1, \ldots, n$ be the vertices of Q and let $S(1), \ldots, S(n)$ be the corresponding simple modules. We choose the order of the vertices in such a way that there is never an arrow in increasing direction, i.e. if $i < j$ in Q_0, then there is no arrow $i \to j$ in Q_1. This can be achieved by labeling a sink of Q with 1, a sink of the remaining quiver after removing 1 with 2 and proceeding up to n. By this reordering of indices, the standard basis vectors of $\mathbb{Z}^n$ are reordered, or equivalently, maps $\mathbb{Z}^n \to \mathbb{Z}^n$ are conjugated by a permuation matrix. Therefore, the spectrum of eigenvectors of Φ_H does not change by this process.

Now let D be the matrix corresponding to the homological bilinear form, i.e. using Proposition 1.2.3 we have

$$D_{ij} = \left\langle \underline{\dim} S(i), \underline{\dim} S(j) \right\rangle = \delta_{ij} - \#\{\alpha \in Q_1 \mid s(\alpha) = i, t(\alpha) = j\}.$$

Therefore, since there are no arrows in increasing direction, D is a lower unitriangular matrix. Using Proposition 2.1.15 and setting $A = D + D^t$, we get $\Phi_H = C(A, \mathrm{id})$. □

Remark 2.1.17. Let C_H be the Cartan matrix of $H = kQ$ with vertices of Q ordered as in the proof of Theorem 2.1.16. We have $D = C_H^{-t}$, where D is the matrix corresponding to the homological bilinear form. Since D is lower unitriangular, we get that $C_H = D^{-t}$ is upper unitriangular. Therefore, the Cartan matrix C_H is not a generalized Cartan matrix. We hope that this is not confusing.

We further remark that clearly $\det(C_H) = 1$, which is a long known special case of the *Cartan determinant Conjecture*, see also [FZH86].

We conclude this section by giving an alternative formulation of the theorem we want to ultimately prove in this chapter. Therefore we need a reminder about properties of matrices:

Definition 2.1.18 (Quadratic form of a symmetric matrix). Let $A \in M^{n\times n}(\mathbb{R})$ symmetric. Then A defines the *quadratic form*

$$q_A : \mathbb{R}^n \to \mathbb{R}, \; x \mapsto x^t A x.$$

We remind the reader that a quadratic form $q : \mathbb{R}^n \to \mathbb{R}$ is called *positive definite* if and only if $q(x) > 0$ for all $0 \neq x \in \mathbb{R}^n$, positive semidefinite if $q(x) \geq 0$ for all $x \in \mathbb{R}^n$ and indefinite if there are both positive and negative values. If Q is a wild quiver, then by definition the associated Tits form q_Q (from now on seen as a quadratic form $\mathbb{R}^n \to \mathbb{R}$ instead of $\mathbb{Z}^n \to \mathbb{Z}$) is indefinite.

Definition 2.1.19 (Definite matrix). We say that a symmetric real matrix A is *positive definite*, *positive semidefinite* or *indefinite* if and only if the associated quadratic form q_A has these properties.

Proposition 2.1.20. *Let $H = kQ$ be a finite-dimensional path algebra of a connected quiver Q. Let A be the corresponding symmetric generalized Cartan matrix, as constructed in the proof of Theorem 2.1.16. Then we have $q_A = 2q_Q$. In particular, Q is Dynkin, Euclidean or wild if and only if A is positive definite, positive semidefinite or indefinite, respectively.*

Proof. By the proof of Theorem 2.1.16 we can assume that the vertices of Q are ordered in such a way that there is no arrow in increasing direction. Then we have $A_{ii} = 2$, $A_{ij} = -\#\{\alpha : i \to j\}$ for $i > j$ and $A_{ij} = A_{ji}$. We get

$$\begin{aligned} q_A(x) = x^t A x &= \sum_{i,j} x_i A_{ij} x_j \\ &= \sum_{i=1}^{n} A_{ii} x_i^2 + \sum_{i \neq j} A_{ij} x_i x_j \\ &= \sum_{i=1}^{n} 2x_i^2 + \sum_{i>j} 2A_{ij} x_i x_j \\ &= 2\left(\sum_{i=1}^{n} x_i^2 - \sum_{\alpha \in Q_1} x_{s(\alpha)} x_{t(\alpha)} \right) \\ &= 2q_Q(x), \end{aligned}$$

proving the claim. □

Definition 2.1.21 (Graph of A, connectedness). Let $A \in M^{n\times n}(\mathbb{R})$ be a symmetric matrix. The unoriented *graph of A* is the graph with vertices $\{1, \ldots, n\}$ and exactly one edge between $i \neq j$ whenever $A_{ij} \neq 0 \neq A_{ji}$. A is called *connected* if its graph is connected.

Clearly, if Q is a connected quiver and A the associated generalized Cartan matrix then A is connected. Therefore, in the next sections we will prove the following theorem, which implies by the preceding discussion (i) and (ii) of Theorem 1.3.1. We will proof statement (iii) afterwards by going back to the original formulation.

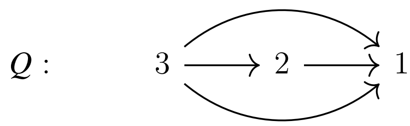
Theorem 2.1.22. *Let A be a connected, indefinite, symmetric generalized Cartan matrix and let $\pi : \{1, \ldots, n\} \to \{1, \ldots, n\}$ be a permutation. Let $C = C(A, \pi)$ be the associated Coxeter transformation. Let ρ be the spectral radius of C. Then we have the following:*

(i) $\rho > 1$ and ρ is itself an eigenvalue of C with algebraic multiplicity one.

(ii) If $\rho \neq \lambda \in \operatorname{spec}(C)$ then $|\lambda| < \rho$.

Remark 2.1.23. In order to avoid confusion by notation we want to say explicitly that the Coxeter transformation $C = C(A, \pi)$ is not the same as the Cartan matrix $C = C_H$ of an algebra H.

Example 2.1.24. Look at the following quiver:



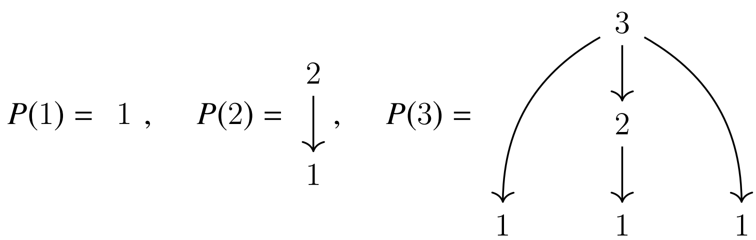
The vertices are already labeled in such a way that there is never an arrow in increasing direction, as in Theorem 2.1.16. Therefore, the corresponding symmetric generalized Cartan matrix looks as follows:

$$A = \begin{pmatrix} 2 & -1 & -2 \\ -1 & 2 & -1 \\ -2 & -1 & 2 \end{pmatrix}$$

We compute the quadratic form q_Q:

$$\begin{aligned} q_Q(x) &= \sum_{i=1}^{3} x_i^2 - \sum_{\alpha \in Q_1} x_{s(\alpha)} x_{t(\alpha)} \\ &= x_1^2 + x_2^2 + x_3^2 - x_1 x_2 - 2x_1 x_3 - x_2 x_3. \end{aligned}$$

We can also clearly check that $q_A = 2q_Q$, as Proposition 2.1.20 says. We have $q_Q(1, 1, 1) = -1 < 0$ and $q_Q(-1, 1, 1) = 5 > 0$ and therefore Q is a wild quiver. A is thus a connected, indefinite, symmetric generalized Cartan matrix. We would like to check that Theorem 2.1.22 holds for the Coxeter transformation $C(A, \mathrm{id})$, which is by Theorem 2.1.16 the same as the Coxeter transformation Φ_H of the algebra $H = kQ$. But let us first check that the two Coxeter transformations really are the same, in order to get faith in the theorems. The indecomposable projective modules look as follows (different vertices denote different basis elements and arrows denote how arrows of the quiver map the basis elements):



Therefore, the dimension vectors of the indecomposable projective modules are

$$\underline{\dim}P(1) = \begin{pmatrix}1\\0\\0\end{pmatrix}, \quad \underline{\dim}P(2) = \begin{pmatrix}1\\1\\0\end{pmatrix}, \quad \underline{\dim}P(3) = \begin{pmatrix}3\\1\\1\end{pmatrix}.$$

The corresponding Cartan matrix is given by

$$C_H = \left(\underline{\dim}P(1), \ \underline{\dim}P(2), \ \underline{\dim}P(3)\right) = \begin{pmatrix}1 & 1 & 3\\0 & 1 & 1\\0 & 0 & 1\end{pmatrix}.$$

Hence, the Coxeter transformation is given by

$$\Phi_H = -C_H^t C_H^{-1} = -\begin{pmatrix}1 & 0 & 0\\1 & 1 & 0\\3 & 1 & 1\end{pmatrix} \cdot \begin{pmatrix}1 & -1 & -2\\0 & 1 & -1\\0 & 0 & 1\end{pmatrix} = \begin{pmatrix}-1 & 1 & 2\\-1 & 0 & 3\\-3 & 2 & 6\end{pmatrix}.$$

If we work instead with the generalized Cartan matrix, we get the same coxeter transformation. Explicitly, the reflections corresponding to A are given by

$$R_1 = \begin{pmatrix}-1 & 1 & 2\\0 & 1 & 0\\0 & 0 & 1\end{pmatrix}, \quad R_2 = \begin{pmatrix}1 & 0 & 0\\1 & -1 & 1\\0 & 0 & 1\end{pmatrix}, \quad R_3 = \begin{pmatrix}1 & 0 & 0\\0 & 1 & 0\\2 & 1 & -1\end{pmatrix}.$$

and we easily compute $C(A, \mathrm{id}) = R_3R_2R_1 = \Phi_H$, as expected.

We now check Theorem 2.1.22 on $C(A, \mathrm{id}) = \Phi_H$: You can compute (for example by using the rule of Sarrus) that the characteristic polynomial is given by

$$\chi_{\Phi_H}(X) = (X+1)(X-\rho)(X-\rho^{-1}),$$

where $\rho = 3 + 2\sqrt{2}$ is the spectral radius of Φ_H. ρ^{-1} is given by $3 - 2\sqrt{2}$. Therefore, the conclusions of Theorem 2.1.22 clearly hold in this situation. We will come back to this example in the end of this chapter.

2.2 There is always an eigenvalue $\lambda > 1$

Before we can prove that $C(A, \pi)$ has always an eigenvalue > 1 we begin by studying the effects of permutation matrices. This will allow us to reduce to the case that $\pi = \mathrm{id}$.

Let $\pi : \{1, \ldots, n\} \to \{1, \ldots, n\}$ be a permutation. Following [Lad08], we define permutation matrices $P^\pi : \mathbb{R}^n \to \mathbb{R}^n$, $e(i) \mapsto e(\pi(i))$ and define for a matrix $A \in M^{n\times n}(\mathbb{R})$ the permuted matrix $A^\pi \in M^{n\times n}(\mathbb{R})$ by $(A^\pi)_{ij} = A_{\pi(i)\pi(j)}$. We show several properties:

Lemma 2.2.1. *For any matrix $A \in M^{n\times n}(\mathbb{R})$ we have $A^\pi = (P^\pi)^{-1} AP^\pi$.*

Proof. We have

$$\begin{aligned}(P^\pi \circ A^\pi)(e(j)) &= P^\pi\left(\sum_{i=1}^n A_{\pi(i)\pi(j)} e(i)\right)\\ &= \sum_{i=1}^n A_{\pi(i)\pi(j)} e(\pi(i))\\ &= \sum_{i=1}^n A_{i\pi(j)} e(i)\\ &= A(e(\pi(j)))\\ &= (A\circ P^\pi)(e(j)),\end{aligned}$$

which proves the claim. □

Lemma 2.2.2. *We have* $(P^\pi)^{-1} = (P^\pi)^t$.

Proof. We have $(P^\pi)^{-1}(e(\pi(j))) = e(j)$. The i'th entry of $(P^\pi)^t(e(\pi(j)))$ is given by

$$\begin{aligned}\left[(P^\pi)^t(e(\pi(j)))\right]_i &= \left((P^\pi)^t\right)_{i\pi(j)} = (P^\pi)_{\pi(j)i}\\ &= [P^\pi(e(i))]_{\pi(j)} = e(\pi(i))_{\pi(j)}\\ &= \delta_{\pi(i)\pi(j)} = \delta_{ij} = e(j)_i,\end{aligned}$$

which finishes the proof. □

Lemma 2.2.3. *Let* $A \in M^{n\times n}(\mathbb{R})$ *be a symmetric matrix. Then* A^π *is also symmetric and for the associated quadratic forms we have*

$$q_{A^\pi} = q_A \circ P^\pi : \mathbb{R}^n \to \mathbb{R}^n \to \mathbb{R}.$$

It follows that A *is indefinite if and only if* A^π *is indefinite.*

Proof. It is clear that A^π is also symmetric. By Lemma 2.2.1 and Lemma 2.2.2 we have

$$\begin{aligned}(q_A\circ P^\pi)(x) = q_A(P^\pi(x)) &= (P^\pi x)^t A(P^\pi x) = x^t\left((P^\pi)^t AP^\pi\right)x\\ &= x^t\left((P^\pi)^{-1}AP^\pi\right)x = x^tA^\pi x = q_{A^\pi}(x).\end{aligned}$$

Now it easily follows that q_A is indefinite if and only if q_{A^π} is indefinite. □

Proposition 2.2.4. *Let* A *be a generalized Cartan matrix and* $\pi : \{1,\ldots,n\} \to \{1,\ldots,n\}$ *be a permutation. Then* A^π *is again a generalized Cartan matrix. We have*

$$C(A,\pi) = P^\pi C(A^\pi, \mathrm{id})(P^\pi)^{-1}.$$

In particular, $C(A,\pi)$ *and* $C(A^\pi,\mathrm{id})$ *are conjugate and therefore have the same spectral properties.*

Proof. That A^π is again a generalized Cartan matrix is clear.

For the statement about the Coxeter transformations, write R_i^A for the i'th reflection corresponding to A and $R_i^{A^\pi}$ for the one corresponding to A^π. Then we get

$$\begin{aligned}\left(P^\pi \circ R_i^{A^\pi}\right)(e(j)) &= P^\pi(e(j) - A^\pi_{ji}e(i)) \\ &= e(\pi(j)) - A_{\pi(j)\pi(i)}e(\pi(i)) \\ &= R^A_{\pi(i)}(e(\pi(j))) \\ &= \left(R^A_{\pi(i)} \circ P^\pi\right)(e(j))\end{aligned}$$

and therefore $R^A_{\pi(i)} = P^\pi \circ R_i^{A^\pi} \circ (P^\pi)^{-1}$. We conclude

$$\begin{aligned}C(A,\pi) &= R^A_{\pi(n)} \circ \cdots \circ R^A_{\pi(1)} \\ &= \left[P^\pi R_n^{A^\pi}(P^\pi)^{-1}\right] \circ \cdots \circ \left[P^\pi R_1^{A^\pi}(P^\pi)^{-1}\right] \\ &= P^\pi \left(R_n^{A^\pi} \circ \cdots \circ R_1^{A^\pi}\right)(P^\pi)^{-1} \\ &= P^\pi C(A^\pi, \mathrm{id})(P^\pi)^{-1},\end{aligned}$$

which finishes the proof. □

In the remainder of this section we follow [How82].

Definition 2.2.5 (Irreducible matrix). A non-negative square matrix $M \in M^{n\times n}(\mathbb{Z})$ is called *irreducible* if for all index pairs $(i,j) \in \{1,\dots,n\}^2$ there exists a natural number $m > 0$ such that

$$(M^m)_{ij} = \sum_{k_1,\dots,k_{m-1}=1}^{n} M_{ik_1}M_{k_1k_2}\cdots M_{k_{m-1}j} > 0.$$

In the following, we don't need the distinction between A_+ and A_- anymore, since we work with symmetric matrices. So we will just write D for A_+ and A_-.

Lemma 2.2.6. *Let $A \in M^{n\times n}(\mathbb{Z})$ with $n \geq 2$ be a connected symmetric generalized Cartan matrix. Let D be the lower triangular matrix of A, i.e.*

$$D_{ij} := \begin{cases} A_{ij}, & i > j \\ 1, & i = j \\ 0, & i < j \end{cases}$$

Then for $0 < \mu$ the matrix

$$P(\mu) = (1+\mu)\,\mathrm{Id} - D^t - \mu D$$

is irreducible.

Proof. Clearly, $P(\mu)$ is non-negative (with zero diagonal). We know that A is connected, i.e. the underlying graph is connected, which means that for indices $i \neq j$ there is a path $i - k_1 - \cdots - k_{m-1} - j$ with pairwise different vertices. That means $A_{ik_1}\cdots A_{k_{m-1}j} \neq 0$. It follows that $P(\mu)_{ik_1}\cdots P(\mu)_{k_{m-1}j} > 0$. We get $(P(\mu)^m)_{ij} > 0$. For $i = j$ we choose any $k \neq i$ which is connected in the graph of A to i (this exists since $n \geq 2$) and get $A_{ik}A_{ki} \neq 0$ and thus $\left(P(\mu)^2\right)_{ii} > 0$. All in all we see that $P(\mu)$ is irreducible. □

The following theorem is a slightly weaker version of [Sen73, Theorem 1.5]:

Theorem 2.2.7 (Perron-Frobenius for irreducible matrices). *Let $M \geq 0$ be an irreducible matrix. Let ρ be the spectral radius of M. Then $\rho > 0$ is an eigenvalue of M with algebraic multiplicity one.*

Proposition 2.2.8. *Let A be a connected, indefinite, symmetric generalized Cartan matrix. Let $\pi : \{1, \ldots, n\} \to \{1, \ldots, n\}$ be a permutation. Then $C(A, \pi)$ has an eigenvalue λ with $\lambda > 1$. In particular for the spectral radius of $C(A, \pi)$ we have $\rho > 1$.*

Proof. By Lemma 2.2.3 we know that A^π is again a symmetric indefinite matrix. Clearly, with A also the matrix A^π is connected. By Proposition 2.2.4 we know that A^π is again a generalized Cartan matrix and that $C(A, \pi)$ is conjugate to $C(A^\pi, \mathrm{id})$. By these considerations we can reduce to the case $\pi = \mathrm{id}$. Let D be the lower triangular matrix of A as in Proposition 2.2.6. Then by Theorem 2.1.13 we know that $C = C(A, \mathrm{id}) = -D^{-1}D^t$.

For $0 \leq \mu \leq 1$ we look at the matrix $P(\mu)$ from Proposition 2.2.6. Let $r(\mu)$ be the spectral radius of $P(\mu)$. We have $r(0) = \rho(\mathrm{Id} - D^t) = 0$ since $\mathrm{Id} - D^t$ is an upper triangular matrix with zero diagonal. We now investigate $r(1)$: Since $A = D + D^t$ is indefinite, it has an eigenvalue $a < 0$ (see for example [Fis13, ch. 5.7.3]). Then there exists $x \neq 0$ such that $A(x) = ax$ and therefore

$$P(1)(x) = \left(2\,\mathrm{Id} - D^t - D\right)(x) = 2x - A(x) = (2 - a)(x),$$

i.e. $2 - a > 2$ is an eigenvalue of $P(1)$. We conclude $r(1) > 2$.

Now we investigate the function

$$f := r - \mathrm{id} : [0, 1] \to \mathbb{R}, \;\; \mu \mapsto r(\mu) - \mu.$$

We have $f(0) = 0$ and $f(1) > 1$. Since f is continuous ($r(\mu)$ depends continuously on the coefficients of the characteristic polynomial of $P(\mu)$, which depend continuously on μ) there is some $\mu \in (0, 1)$ such that $f(\mu) = 1$, i.e. $r(\mu) = 1 + \mu$. We fix this μ. $P(\mu)$ is irreducible by Lemma 2.2.6 (note that $n \geq 2$ holds since otherwise A could not be indefinite) and therefore $r(\mu)$ is an eigenvalue of $P(\mu)$ by Theorem 2.2.7. Therefore we conclude

$$\begin{aligned} 0 = \det(r(\mu)\,\mathrm{Id} - P(\mu)) &= \det\left((1 + \mu)\,\mathrm{Id} - (1 + \mu)\,\mathrm{Id} + D^t + \mu D\right) \\ &= \det\left(D^t + \mu D\right) = \det((1/\mu)D^t + D) = \det\left((1/\mu)D + D^t\right) \\ &= \det\left((1/\mu)\,\mathrm{Id} - (-D^{-1}D^t)\right) = \det((1/\mu)\,\mathrm{Id} - C(A, \mathrm{id})), \end{aligned}$$

where many of the steps make sense precisely because the determinants are all zero. This shows that $\lambda := 1/\mu > 1$ is an eigenvalue of $C(A, \mathrm{id})$. □

2.3 The quiver of (A, π) and admissible changes

In this section we further investigate what happens with the Coxeter transformation $C(A, \pi)$ if we change the permutation π. It turns out that there are certain *admissible* *changes* that don't change the conjugacy class of the Coxeter transformation and thus don't change the spectral properties. We follow [Rin94].

First we alter the notation a little bit: We define $I := \{1, \ldots, n\}$ and write $\pi : \{1, \ldots, n\} \to I$ for a permutation. This will make thinking about Coxeter transformations easier since by writing $\{1, \ldots, n\}$ or I it is already clear if we talk about the domain or the codomain of the permutation π. In fact, we don't even need I to be the set $\{1, \ldots, n\}$ and would be well advised to think about it as any set with n elements that *gets ordered* by a bijection $\pi : \{1, \ldots, n\} \to I$ (i.e. we forget completely that I has a default order). We fix a (not necessarily symmetric) generalized Cartan matrix $A \in \mathbb{R}^{I \times I}$. Thus $A_{ii} = 2$, $A_{ij} \leq 0$ for all $i \neq j$ and $A_{ij} = 0$ if and only if $A_{ji} = 0$ for $i \neq j$.

For clarity, we make some definitions explicit again: The real vector space $\mathbb{R}^I$ has a canonical basis $(e(i))_{i \in I}$. A matrix $B = \left(B_{ij}\right)_{i,j \in I}$ can then be seen as a linear map $B : \mathbb{R}^I \to \mathbb{R}^I$, $e(j) \mapsto \sum_{i \in I} B_{ij} e(i)$. The i-th reflection associated to A is defined as the linear map $R_i : \mathbb{R}^I \to \mathbb{R}^I$ satisfying

$$R_i(e(j)) = e(j) + \alpha_{ji} e(i),$$

where by definition $\alpha_{ji} = -A_{ji}$.

Definition 2.3.1 (Coxeter transformation). Let $\pi : \{1, \ldots, n\} \to I$ be a bijection. Then we define the *Coxeter transformation for A* (with respect to π) to be the composition

$$C = C(A, \pi) : \mathbb{R}^I \to \mathbb{R}^I, \ x \mapsto R_{\pi(n)} \cdots R_{\pi(1)} x.$$

We want to define a quiver corresponding to (A, π) such that A, together with the quiver Q, contains all information of the Coxeter transformation.

Definition 2.3.2 (Quiver of a generalized Cartan matrix). Let $\pi : \{1, \ldots, n\} \to I$. Then we define the *quiver for A*, $Q(A, \pi)$, to be the quiver with vertex set I and exactly one arrow $x \to y$ if an only if both $\alpha_{xy} \neq 0$ and $\pi^{-1}(x) < \pi^{-1}(y)$.

This definition already seems fruitful through the following Lemma:

Lemma 2.3.3. *Let $y \neq y' \in Q_0$, where $Q = Q(A, \pi)$. Then R_y and $R_{y'}$ commute if and only if there is no arrow between y and y'.*

Proof. We have

$$R_y R_{y'} e(x) = R_y \left(e(x) + \alpha_{xy'} e(y')\right) = e(x) + \alpha_{xy} e(y) + \alpha_{xy'} \left(e(y') + \alpha_{y'y} e(y)\right)$$

and

$$R_{y'} R_y e(x) = R_{y'} \left(e(x) + \alpha_{xy} e(y)\right) = e(x) + \alpha_{xy'} e(y') + \alpha_{xy} \left(e(y) + \alpha_{yy'} e(y')\right).$$

If $\alpha_{yy'} = \alpha_{y'y} = 0$ (i.e. there is no arrow between y and y'), then those two expressions are equal, so R_y and $R_{y'}$ commute. If those expressions are equal, then we get

$$\alpha_{xy} + \alpha_{xy'} \alpha_{y'y} = \alpha_{xy} \text{ and } \alpha_{xy'} = \alpha_{xy'} + \alpha_{xy} \alpha_{yy'}$$

for all $x \in Q_0$. Plugging in $x = y$ and $x = y'$ shows that $\alpha_{yy'} = \alpha_{y'y} = 0$, i.e. there is no arrow between y and y'. □

Definition 2.3.4 (Reflection at a vertex). Let Q be a quiver and $x \in Q_0$. Then we define the new quiver $\sigma_x Q$ as the quiver with vertex set Q_0 and the same arrows as Q except that all arrows starting or ending in x get reversed (for example if $a : x \to y$ in Q then we get an arrow $a^* : y \to x$ in $\sigma_x Q$).

Definition 2.3.5 (Source sequence). Let Q be a quiver. A sequence $(x_1, \dots, x_m)$ in Q_0 is called a *source sequence* if for all $i \in \{1, \dots, m\}$, x_i is a source in $\sigma_{x_{i-1}} \cdots \sigma_{x_1} Q$.

Definition 2.3.6 (Admissible change of orientation). Let $(x_1, \dots, x_m)$ be a source sequence in Q. Then the product $\omega = \sigma_{x_m} \cdots \sigma_{x_1}$ is called *admissible change of orientation.*

Lemma 2.3.7. *Let $Q = Q(A, \pi)$ and let ω be an admissible change of orientation. Then $\omega Q = Q(A, \pi')$ for a suitable bijection $\pi' : \{1, \dots, n\} \to I$.*

Proof. By induction, we only need to show that if x is a source in Q, then there exists $\pi' : \{1, \dots, n\} \to I$ such that $\sigma_x Q = Q(A, \pi')$: Write $x = \pi(i)$. Then we define

$$\pi' : \{1, \dots, n\} \to I, \ j \mapsto \begin{cases} \pi(j), \ 1 \leq j < i \\ \pi(j+1), \ i \leq j < n \\ x, \ j = n \end{cases}$$

This yields the result. □

Proposition 2.3.8. *Let $Q = Q(A, \pi)$. Let ω be an admissible change of orientation and write $\omega Q = Q(A, \pi')$ as in Proposition 2.3.7. Then the Coxeter transformations $C(A, \pi)$ and $C(A, \pi')$ are conjugate. In particular they have the same spectral properties.*

Proof. Since being similar is an equivalence relation, we only need to show this for $\omega = \sigma_x$ where $x = \pi(i)$ is a source in Q. Then let π' be the bijection constructed in the proof of Lemma 2.3.7. We get $C(A, \pi) = R_{\pi(n)} \cdots R_{\pi(1)}$, which is by Lemma 2.3.3 the same as $R_{\pi(n)} \cdots R_{\pi(i+1)} R_{\pi(i-1)} \cdots R_{\pi(1)} R_{\pi(i)}$. By setting $S := R_{\pi(i)}$ we see $C(A, \pi') = S \cdot C(A, \pi) \cdot S^{-1}$. □

2.4 The tree case

In this section we proof Theorem 2.1.22 in the case that the quiver $Q(A, \pi)$ is a tree, by which we mean that it is connected and the underlying graph does not have any circuits of length ≥ 1. In doing so we follow [dlP94], which bases its computations on [A'C76]. We remark that we use at many places conventions for signs and names of matrices which are different from those of the original articles.

We begin by proving that in the tree case there is always an admissible change of orientation such that the quiver has a *sink-source orientation*:

Definition 2.4.1 (Sink-source oriented). Let Q be a quiver. It is *sink-source-oriented* in case that every vertex in Q is a sink or a source.

Proposition 2.4.2. *Let Q be a quiver which is a tree. Then there is an admissible change of orientation ω such that ωQ is in sink-source orientation.*

Proof. We do this by induction on the number of arrows in Q. If there is no arrow, then Q is only one vertex and thus clearly in sink-source orientation. Now let Q have at least one arrow and let α be an arrow between x and y such that one of x and y is a leaf (i.e. α is the only arrow connecting to it). Without loss of generality, y is the leaf. Consider the quiver Q' obtained from Q by deleting α and y. By induction, there is an admissible change of orientation ω' such that $\omega' Q'$ is in sink-source orientation. The same change is not necessarily admissible when applied to Q, since if σ_x is a reflection appearing in ω', it is only admissible in Q if x is at that stage also a source in Q. If not, then we replace σ_x by $\sigma_x \sigma_y$ and thus obtain an admissible change ω such that ωQ is in sink-source orientation. □

We fix from now on in this section a symmetric, indefinite, generalized Cartan matrix A such that its graph (i.e. the graph with vertex set I and an edge between $i \neq j$ if $A_{ji} = A_{ij} \neq 0$) is a tree. Note that for any bijection $\pi : \{1, \ldots, n\} \to I$ the underlying graph of $Q(A, \pi)$ is precisely the graph of A and thus $Q(A, \pi)$ is a tree.

Proposition 2.4.3. *Let $\pi : \{1, \ldots, n\} \to I$ be any permutation. Then the Coxeter transformation $C(A, \pi)$ is conjugate to $C(A, \pi')$ for a bijection π' with the following property:*

There is a number $m \in \{1, \ldots, n\}$ such that for all $1 \leq i \leq m$, $\pi'(i)$ is a source and for all $m+1 \leq i \leq n$, $\pi'(i)$ is a sink in $Q' = Q(A, \pi')$.

Proof. Let $Q = Q(A, \pi)$. Then by Proposition 2.4.2 there is an admissible change of orientation ω such that ωQ is in sink-source orientation. By Lemma 2.3.7 we have $\omega Q = Q(A, \pi') = Q'$ for some bijection $\pi' : \{1, \ldots, n\} \to I$ and by Proposition 2.3.8, the Coxeter transformations $C(A, \pi)$ and $C(A, \pi')$ are conjugate. Consider the case that there are $x = \pi'(i), y = \pi'(j) \in Q'_0$ such that $i = j + 1$ and x is a source and y is a sink. If there was an arrow between x and y in Q' then it would go in the direction $x \to y$ since x is a source. But that is not possible since $i > j$ and thus there is no arrow between x and y. It follows that R_x and R_y commute by Lemma 2.3.3. Therefore π' can be changed in such a way that $\pi'(j) = x$ and $\pi'(i) = y$ without changing the Coxeter transformation. If we do this consecutively for all such pairs i, j then we obtain a number m such that $\pi'(1), \ldots, \pi'(m)$ are the sources in Q' and $\pi'(m+1), \ldots, \pi'(n)$ are the sinks. □

Now let $\pi : \{1, \ldots, n\} \to I$ be a bijection. We want to show Theorem 2.1.22 for the Coxeter transformation $C(A, \pi)$. By Proposition 2.4.3 we can without loss of generality assume that $\{\pi(1), \ldots, \pi(m)\}$ are the sources of $Q = Q(A, \pi)$ and $\{\pi(m+1), \ldots, \pi(n)\}$ are the sinks. To simplify the notation we write from now on i instead of $\pi(i)$. Then we have $C = C(A, \pi) = C(A, \mathrm{id}) = R_n \cdots R_1$.

Remark 2.4.4. If the equality $C(A, \pi) = C(A, \mathrm{id})$ is confusing to you as a reader, then it might be because you still think of I as a set with a default order. Try to think of I as an unordered set that gets its ordering *through* π.

We investigate further how this Coxeter transformation looks like: Let D be again the lower triangular part of A, i.e. $D_{ii} = 1$ and $D_{ij} = A_{ij}$ for all $i > j$, whereas for $i < j$ we have $D_{ij} = 0$. Then we know from Theorem 2.1.13 that $C = -D^{-1}D^t$. We define $N = D - \mathrm{Id}$. Then we get the following new description:

Lemma 2.4.5. *We have $N^2 = 0$ and $C = -(\mathrm{Id} - N)(\mathrm{Id} + N^t)$.*

Proof. We have $N_{ij} = 0$ for all $i \leq j$. Let $i > j$. If $1 \leq j < i \leq m$, then we also have $N_{ij} = A_{ij} = 0$ since i and j both are sources in Q and therefore have no arrow between them. In the same way, for $m+1 \leq j < i \leq n$, we have $N_{ij} = 0$. Therefore, the only nonzero block of N is the $\{m+1,\ldots,n\} \times \{1,\ldots,m\}$-block. It follows $N^2 = 0$. We have $C = -D^{-1}D^t = -D^{-1}(N + \mathrm{Id})^t = -D^{-1}(\mathrm{Id} + N^t)$. It remains to show that $D^{-1} = \mathrm{Id} - N$: $D(\mathrm{Id} - N) = (\mathrm{Id} + N)(\mathrm{Id} - N) = \mathrm{Id} - N + N - N^2 = \mathrm{Id}$. This finishes the proof. □

Lemma 2.4.6. *Let χ_C be the characteristic polynomial of C and χ_{N+N^t} the characteristic polynomial of $N + N^t$. Then we have*

$$\chi_C(X^2) = X^n \chi_{N+N^t}(X + X^{-1}).$$

Proof. We know by Lemma 2.4.5 that $C = -(\mathrm{Id} - N)(\mathrm{Id} + N^t)$. Then we compute:

$$\begin{aligned}
\chi_C(X^2) &= \det\left(X^2\,\mathrm{Id} - C\right) \\
&= \det\left(X^2\,\mathrm{Id} + (\mathrm{Id} - N)(\mathrm{Id} + N^t)\right) \cdot 1 \\
&= \det\left(X^2\,\mathrm{Id} + (\mathrm{Id} - N)(\mathrm{Id} + N^t)\right) \cdot \det\left(\mathrm{Id} - N^t\right) \\
&= \det\left((X^2+1)\,\mathrm{Id} - N - X^2 N^t\right) \\
&= X^n \det\left((X + X^{-1})\,\mathrm{Id} - X^{-1}N - X N^t\right) \\
&= X^n \det\left((X + X^{-1})\,\mathrm{Id} - (N + N^t)\right) \\
&= X^n \chi_{N+N^t}(X + X^{-1}),
\end{aligned}$$

where the individual steps are proven as follows: In the third step we used that $\mathrm{Id} - N^t$ is an upper triangular matrix which only has 1's on its diagonal and thus has determinant 1. For the fourth step we remember that $N^2 = 0$ by Lemma 2.4.5 (and therefore also $(N^t)^2 = 0$) and compute

$$\begin{aligned}
\left(X^2\,\mathrm{Id} + (\mathrm{Id} - N)(\mathrm{Id} + N^t)\right) \cdot (\mathrm{Id} - N^t) &= \left(X^2\,\mathrm{Id} + (\mathrm{Id} - N)(\mathrm{Id} + N^t)\right) - X^2 N^t - (\mathrm{Id} - N)\,N^t \\
&= X^2\,\mathrm{Id} + \mathrm{Id} + N^t - N - NN^t - X^2 N^t - N^t + NN^t \\
&= (X^2+1)\,\mathrm{Id} - N - X^2 N^t.
\end{aligned}$$

The sixth step can be seen using the Leibniz formula or alternatively as follows: Let M be the $\{m+1,\ldots,n\} \times \{1,\ldots,m\}$-block of N. Then we have

$$\begin{aligned}
&\begin{pmatrix} \mathrm{Id} & 0 \\ 0 & X\,\mathrm{Id} \end{pmatrix} \cdot \begin{pmatrix} (X + X^{-1})\,\mathrm{Id} & -XM^t \\ -X^{-1}M & (X + X^{-1})\,\mathrm{Id} \end{pmatrix} \cdot \begin{pmatrix} \mathrm{Id} & 0 \\ 0 & X^{-1}\,\mathrm{Id} \end{pmatrix} \\
= &\begin{pmatrix} (X + X^{-1})\,\mathrm{Id} & -M^t \\ -M & (X + X^{-1})\,\mathrm{Id} \end{pmatrix} = (X + X^{-1})\,\mathrm{Id} - (N + N^t)
\end{aligned}$$

and the result follows since the matrix in the middle of the upper row is just $(X + X^{-1})\,\mathrm{Id} - X^{-1}N - XN^t$ and since the determinants of the left and the right matrix are inverse to each other. This finishes the proof. □

Lemma 2.4.7. *For any square matrix M we have*

$$\chi_M(-X) = (-1)^n \chi_{-M}(X)$$

Proof. We have

$$\begin{aligned}\chi_M(-X) &= \det(-X\,\mathrm{Id} - M)\\ &= (-1)^n \det(X\,\mathrm{Id} - (-M))\\ &= (-1)^n \chi_{-M}(X).\end{aligned}$$

□

From now on, define B as the matrix $B = -(N + N^t)$:

Lemma 2.4.8. *We have*

$$\chi_C(X^2) = X^n \chi_B(X + X^{-1}).$$

Proof. We have

$$\begin{aligned}\chi_B(X + X^{-1}) &= \chi_{-(N+N^t)}(X + X^{-1})\\ &= (-1)^n \chi_{N+N^t}((-X) + (-X)^{-1}),\end{aligned}$$

where we used Lemma 2.4.7. From this we conclude using Lemma 2.4.6

$$\begin{aligned}\chi_C(X^2) &= \chi_C((-X)^2)\\ &= (-X)^n \chi_{N+N^t}((-X) + (-X)^{-1})\\ &= X^n \chi_B(X + X^{-1}).\end{aligned}$$

□

Proposition 2.4.9. *With the same notation as before we have the following:*

(i) Let $0 \neq \lambda \in \mathbb{C}$. *Then* $\lambda^2 \in \operatorname{spec}(C)$ *if and only if* $\lambda + \lambda^{-1} \in \operatorname{spec}(B)$.

(ii) We have $\operatorname{spec}(C) \subset S^1 \cup \mathbb{R}_{>0}$, *where* $S^1 \subset \mathbb{C}$ *is the standard unit circle.*

Proof. For $\lambda \neq 0$ in $\mathbb{C}$ we have

$$\chi_C\left(\lambda^2\right) = \lambda^n \chi_B\left(\lambda + \lambda^{-1}\right)$$

by Lemma 2.4.8 and therefore get that $\chi_C\left(\lambda^2\right) = 0$ (i.e. $\lambda^2 \in \operatorname{spec}(C)$) if and only if $\chi_B\left(\lambda + \lambda^{-1}\right) = 0$ (i.e. $\lambda + \lambda^{-1} \in \operatorname{spec}(B)$). This proves (i).

Now we prove (ii): Let $p \in \operatorname{spec}(C)$. Since C is invertible (We have $R_i^2 = \mathrm{Id}$ for all reflections R_i) we have $p \neq 0$. Let $\lambda \neq 0$ be one of the two square-root of p, i.e. $\lambda^2 = p$ (this exists since $\mathbb{C}$ is algebraically closed). Then $\lambda^2 \in \operatorname{spec}(C)$ and therefore by (i) $\lambda + \lambda^{-1} \in \operatorname{spec}(B)$. Since B is a symmetric real matrix we have $\operatorname{spec}(B) \subseteq \mathbb{R}$ and therefore $\lambda + \lambda^{-1} \in \mathbb{R}$. Write $\lambda = a + ib$ with $a, b \in \mathbb{R}$. Then we get

$$\lambda + \lambda^{-1} = (a + ib) + \left(\frac{a}{|\lambda|} - i\frac{b}{|\lambda|}\right) = \left(a + \frac{a}{|\lambda|}\right) + i\left(b - \frac{b}{|\lambda|}\right) \in \mathbb{R}$$

and therefore $b = \frac{b}{|\lambda|}$, i.e. $b = 0$ or $|\lambda| = 1$. In the first case we have $0 \neq \lambda \in \mathbb{R}$ and therefore $p = \lambda^2 \in \mathbb{R}_{>0}$, in the second case we get $p = \lambda^2 \in S^1$, which finishes the proof. □

Proof of Theorem 2.1.22 in the tree case. We already know that $\mathrm{spec}(C) \subseteq S^1 \cup \mathbb{R}_{>0}$ by Proposition 2.4.9 (ii). We also know that C has an eigenvalue > 1 by Proposition 2.2.8. Therefore the spectral radius ρ satisfies $\rho > 1$ and is itself an eigenvalue, since the eigenvalue with biggest absolute value must lie in $\mathbb{R}_{>1}$. This also clearly proves that for $\rho \neq \lambda \in \mathrm{spec}(C)$ we have $|\lambda| < \rho$ and therefore (ii) of the theorem.

It remains to show that the algebraic multiplicity of ρ is 1, i.e. ρ is a simple root of χ_C. In order to achieve this we must show $\chi_C'(\rho) \neq 0$. We therefore go on by investigating derivatives more closely. Using Lemma 2.4.8 we have

$$\begin{aligned} 2X\chi_C'(X^2) &= \frac{d}{dX}\left(\chi_C\left(X^2\right)\right) \\ &= \frac{d}{dX}\left(X^n\chi_B(X+X^{-1})\right) \\ &= nX^{n-1}\chi_B(X+X^{-1}) + X^n(1-X^{-2})\chi_B'(X+X^{-1}). \end{aligned}$$

We have $\rho > 1$ and can therefore write $\rho = \lambda^2$ with some $\lambda > 1$. Then $\lambda+\lambda^{-1} \in \mathrm{spec}(B)$. Since ρ is the biggest eigenvalue of C it follows easily that $\lambda+\lambda^{-1}$ is the biggest positive eigenvalue of B (we can write every eigenvalue of B which is > 2 as $\lambda' + \lambda'^{-1}$ for some $\lambda' > 1$ and use that the function $x \mapsto x + x^{-1}$ is strictly increasing for $x > 1$). Now $B = P(1)$ in the notation of Lemma 2.2.6 and therefore B is irreducible and it follows from the Perron-Frobenius Theorem 2.2.7 that $\rho(B)$ is a simple eigenvalue of B. We therefore have $\lambda + \lambda^{-1} = \rho(B)$, $\chi_B(\lambda+\lambda^{-1}) = 0$ and $\chi_B'(\lambda+\lambda^{-1}) \neq 0$ and conclude:

$$\begin{aligned} \chi_C'(\rho) &= \chi_C'(\lambda^2) \\ &= \frac{n\lambda^{n-1}\chi_B(\lambda+\lambda^{-1}) + \lambda^n(1-\lambda^{-2})\chi_B'(\lambda+\lambda^{-1})}{2\lambda} \\ &= \frac{\lambda^{n-1}-\lambda^{n-3}}{2}\chi_B'(\lambda+\lambda^{-1}) \neq 0, \end{aligned}$$

which finishes the proof. □

2.5 The tree case - another proof

In this section we investigate how to prove the tree case by computing the eigenvalues and eigenvectors of C explicitly. These investigations will not be used later. We include them mainly because some of the eigenvector computations in [dlP94] – which are based on [SS78] – are incorrect and since the full proof based on these computations was probably not stated completely in the literature before.

As in the preceding section, we can write $C = -(\mathrm{Id} - N)(\mathrm{Id} + N^t)$. Let M be the $\{m+1,\dots,n\} \times \{1,\dots,m\}$-block of N, as in the proof of Lemma 2.4.6. Then we have

$$\begin{aligned} C &= -(\mathrm{Id} - N)(\mathrm{Id} + N^t) \\ &= -\begin{pmatrix} \mathrm{Id} & 0 \\ -M & \mathrm{Id} \end{pmatrix} \cdot \begin{pmatrix} \mathrm{Id} & M^t \\ 0 & \mathrm{Id} \end{pmatrix} \end{aligned}$$

$$= -\begin{pmatrix} \mathrm{Id} & M^t \\ -M & -MM^t + \mathrm{Id} \end{pmatrix}$$
$$= \begin{pmatrix} -\mathrm{Id} & -M^t \\ M & MM^t - \mathrm{Id} \end{pmatrix}$$

The matrix $E := M^tM \in M^{m\times m}(\mathbb{R})$ is real symmetric. The spectral Theorem [Fis13, ch. 5.6] guarantees that there is an orthonormal basis $\{x_1, \dots, x_m\}$ of $\mathbb{R}^m$ consisting of eigenvectors of E. Let $v_1, \dots, v_m$ be the corresponding eigenvalues. Let q_E be the quadratic form of E. Then we get

$$q_E(x) = x^tEx = x^tM^tMx = (Mx)^t(Mx) = \langle Mx, Mx\rangle = \|Mx\|^2 \geq 0,$$

i.e. E is positive semidefinite. This shows that $v_i \geq 0$ for all i (see for example again [Fis13, ch. 5.7.3]). The idea is now to use the x_i and the v_i to construct eigenvectors and eigenvalues of C (or in fact the Jordan normal form of C) explicitly. We do this as follows:

We order the eigenvalues in such a way that $v_i \neq 0, 4$ for all $1 \leq i \leq p$, that $v_i = 4$ for all $p+1 \leq i \leq q$ and that $v_i = 0$ for all $q+1 \leq i \leq m$. Therefore we have $\mathrm{rank}(E) = q$.

Lemma 2.5.1. *We have* $\dim\ker(M^t) = n - m - q$.

Proof. For any real matrix Q we have $\ker(Q) = \ker(Q^tQ)$ since if $Q^tQ(x) = 0$ we get

$$0 = x^tQ^tQx = (Qx)^t(Qx) = \langle Qx, Qx\rangle = \|Qx\|^2$$

and therefore $Qx = 0$. Therefore we get

$$\mathrm{rank}(M^t) = \mathrm{rank}(M) = \mathrm{rank}(M^tM) = \mathrm{rank}(E) = q,$$

so $\dim\ker(M^t) = n - m - q$. □

Let $\{x'_{q+1}, \dots, x'_{n-m}\}$ be a basis of $\ker(M^t)$. Then we construct the eigenvalues and eigenvectors of C as follows:

For $1 \leq i \leq p$ we set

$$\lambda_{i1} = \frac{1}{2}v_i - 1 + \frac{1}{2}\sqrt{v_i(v_i-4)}, \quad \lambda_{i2} = \frac{1}{2}v_i - 1 - \frac{1}{2}\sqrt{v_i(v_i-4)}$$

For $p+1 \leq i \leq q$ we just define $\lambda_{ij} = 1$ for $j = 1, 2$. We further define for $1 \leq i \leq q$

$$y_{ij} = \begin{pmatrix} x_i \\ b_{ij}Mx_i \end{pmatrix},$$

where the b_{ij} are defined as follows:

$$b_{ij} = \begin{cases} -\frac{1+\lambda_{ij}}{v_i}, & 1 \leq i \leq p \\ -\frac{1}{2}, & p+1 \leq i \leq q \text{ and } j = 1 \\ -\frac{3}{4}, & p+1 \leq i \leq q \text{ and } j = 2 \end{cases}$$

Furthermore we set

$$\lambda_{ij} = -1 \text{ for } \begin{cases} q+1 \le i \le m \text{ and } j=1 \\ q+1 \le i \le n-m \text{ and } j=2 \end{cases}$$

and

$$y_{ij} = \begin{cases} \begin{pmatrix} x_i \\ 0 \end{pmatrix}, & q+1 \le i \le m \text{ and } j=1 \\ \begin{pmatrix} 0 \\ x'_i \end{pmatrix}, & q+1 \le i \le n-m \text{ and } j=2 \end{cases}$$

Lemma 2.5.2. *The set* $\{Mx_1, \dots, Mx_q, x'_{q+1}, \dots, x'_{n-m}\}$ *is a basis of* $\mathbb{R}^{n-m}$.

Proof. We only need to show they are linearly independent. Let $\mu_1, \dots, \mu_{n-m} \in \mathbb{R}$ satisfy the equation

$$\sum_{i=1}^{q} \mu_i M x_i + \sum_{i=q+1}^{n-m} \mu_i x'_i = 0.$$

Since the x'_i are in $\ker M^t$ we get

$$0 = M^t \left(\sum_{i=1}^{q} \mu_i M x_i + \sum_{i=q+1}^{n-m} \mu_i x'_i \right) = \sum_{i=1}^{q} \mu_i E x_i = \sum_{i=1}^{q} \mu_i v_i x_i.$$

Now since $\{x_1, \dots, x_q\}$ is linearly independent we get $\mu_i v_i = 0$ for all $i = 1, \dots, q$. Since all these v_i are nonzero by construction we conclude $\mu_1 = \cdots = \mu_q = 0$. Since $\{x'_{q+1}, \dots, x'_{n-m}\}$ is linearly independent by definition we deduce that also $\mu_{q+1} = \cdots = \mu_{n-m} = 0$, finishing the proof. □

Proposition 2.5.3. *These constructions have the following properties:*

(i) $\{y_{11}, y_{12}, \dots, y_{q1}, y_{q2}, y_{q+1,1}, \dots, y_{m1}, y_{q+1,2}, \dots, y_{n-m,2}\}$ *is a basis of* $\mathbb{R}^n$.

(ii) C, respresented in the basis of (i), is in Jordan normal form. More precisely we have the following:

- *(a) For* $1 \le i \le p$ *and* $j = 1, 2$ *we have* $Cy_{ij} = \lambda_{ij} y_{ij}$.
- *(b) For* $p+1 \le i \le q$ *we have* $Cy_{i1} = \lambda_{i1} y_{i1} = y_{i1}$ *and* $Cy_{i2} = y_{i1} + \lambda_{i2} y_{i2} = y_{i1} + y_{i2}$.
- *(c) For* $i \ge q+1$ *we have* $Cy_{ij} = \lambda_{ij} y_{ij} = -y_{ij}$.

Proof. Clearly, the y_{ij} are precisely $2q + (m-q) + (n-m-q) = n$ vectors, so there is a chance they form a basis of $\mathbb{R}^n$. We prove this by showing that they are linearly independent. Let $a_{ij} \in \mathbb{R}$ such that

$$\sum_{i=1}^{q} (a_{i1} y_{i1} + a_{i2} y_{i2}) + \sum_{i=q+1}^{m} a_{i1} y_{i1} + \sum_{i=q+1}^{n-m} a_{i2} y_{i2} = 0.$$

Evaluating the upper m entries and the lower $n-m$ entries of this equation seperately, we get the two equations

$$\sum_{i=1}^{q}(a_{i1}+a_{i2})x_i+\sum_{i=q+1}^{m}a_{i1}x_i=0, \tag{2.1}$$

$$\sum_{i=1}^{q}(a_{i1}b_{i1}+a_{i2}b_{i2})Mx_i+\sum_{i=q+1}^{n-m}a_{i2}x'_i=0. \tag{2.2}$$

Since $\{x_1,\dots,x_m\}$ is a basis of $\mathbb{R}^m$ we get from 2.1 that

$$\begin{aligned} a_{i1}+a_{i2}&=0 \text{ for all } i\in\{1,\dots,q\},\\ a_{i1}&=0 \text{ for all } i\in\{q+1,\dots,m\}, \end{aligned} \tag{2.3}$$

and since by Lemma 2.5.2 the set $\{Mx_1,\dots,Mx_q,x'_{q+1},\dots,x'_{n-m}\}$ is a basis of $\mathbb{R}^{n-m}$ we conclude from 2.2 that

$$\begin{aligned} a_{i1}b_{i1}+a_{i2}b_{i2}&=0 \text{ for all } i\in\{1,\dots,q\},\\ a_{i2}&=0 \text{ for all } i\in\{q+1,\dots,n-m\}. \end{aligned} \tag{2.4}$$

The relations 2.3 and 2.4 together show that

$$a_{i1}(b_{i1}-b_{i2})=0 \text{ for all } i\in\{1,\dots,q\},$$

which is - since $b_{i1}\neq b_{i2}$ - only possible if $a_{11}=\dots=a_{q1}=0$. Then 2.3 shows that also $a_{12}=\dots=a_{q2}=0$. All in all, we showed that $a_{ij}=0$ for all i,j and so we indeed found a basis. This proves (i).

For (ii), let $i\in\{1,\dots,q\}$ and $j\in\{1,2\}$. Then we get

$$Cy_{ij}=\begin{pmatrix}-\mathrm{Id} & -M^t\\ M & MM^t-\mathrm{Id}\end{pmatrix}\cdot\begin{pmatrix}x_i\\ b_{ij}Mx_i\end{pmatrix}=\begin{pmatrix}(-1-b_{ij}v_i)x_i\\ (1+b_{ij}v_i-b_{ij})Mx_i\end{pmatrix}. \tag{2.5}$$

Now consider the more special case $i\in\{1,\dots,p\}$. Then we have $b_{ij}=-\frac{1+\lambda_{ij}}{v_i}$ and therefore

$$Cy_{ij}=\begin{pmatrix}\lambda_{ij}x_i\\ \left(1-(1+\lambda_{ij})+\frac{1+\lambda_{ij}}{v_i}\right)Mx_i\end{pmatrix}.$$

Thus, in order to show that $Cy_{ij}=\lambda_{ij}y_{ij}$ we must show that $-\lambda_{ij}+\frac{1+\lambda_{ij}}{v_i}=\lambda_{ij}b_{ij}$, which is by definition equal to $-\lambda_{ij}\frac{1+\lambda_{ij}}{v_i}$. This follows easily from the fact that $\lambda_{ij}=\frac{1}{2}v_i-1\pm\frac{1}{2}\sqrt{v_i(v_i)-4}$ by definition.

Next consider the case $i\in\{p+1,\dots,q\}$ and $j=1$. Then $b_{ij}=-\frac{1}{2}$ and $v_i=4$ and therefore

$$Cy_{i1}=\begin{pmatrix}(-1+\frac{1}{2}\cdot 4)x_i\\ (1-\frac{1}{2}\cdot 4+\frac{1}{2})Mx_i\end{pmatrix}=\begin{pmatrix}x_i\\ -\frac{1}{2}Mx_i\end{pmatrix}=y_{i1}=\lambda_{i1}y_{i1}$$

Now we consider the case that $i\in\{p+1,\dots,q\}$ and $j=2$. Then $b_{ij}=-\frac{3}{4}$ and still $v_i=4$ and we get

$$Cy_{i2}=\begin{pmatrix}2x_i\\ -\frac{5}{4}Mx_i\end{pmatrix}=\begin{pmatrix}x_i\\ -\frac{1}{2}Mx_i\end{pmatrix}+\begin{pmatrix}x_i\\ -\frac{3}{4}Mx_i\end{pmatrix}=y_{i1}+y_{i2}=y_{i1}+\lambda_{i2}y_{i2}.$$

Next we consider the case $i \in \{q+1, \ldots, m\}$ and $j = 1$. Then

$$Cy_{i1} = \begin{pmatrix} -\mathrm{Id} & -M^t \\ M & MM^t - \mathrm{Id} \end{pmatrix} \cdot \begin{pmatrix} x_i \\ 0 \end{pmatrix} = \begin{pmatrix} -x_i \\ Mx_i \end{pmatrix} = -\begin{pmatrix} x_i \\ 0 \end{pmatrix} = -y_{i1} = \lambda_{i1} y_{i1},$$

where we used that $Mx_i = 0$, which follows from $Ex_i = v_i x_i = 0$ and the computation

$$0 = x_i^t E x_i = x_i^t M^t M x_i = \langle Mx_i, Mx_i \rangle = \|Mx_i\|^2.$$

Finally let $i \in \{q+1, \ldots, n-m\}$ and $j = 2$. Then we have $M^t x_i' = 0$ by definition of x_i' and therefore

$$Cy_{i2} = \begin{pmatrix} -\mathrm{Id} & -M^t \\ M & MM^t - \mathrm{Id} \end{pmatrix} \cdot \begin{pmatrix} 0 \\ x_i' \end{pmatrix} = \begin{pmatrix} -M^t x_i' \\ MM^t x_i' - x_i' \end{pmatrix} = -\begin{pmatrix} 0 \\ x_i' \end{pmatrix} = \lambda_{i2} y_{i2}.$$

This finishes the proof. □

Proposition 2.5.4. *We have* $\mathrm{spec}(C) \subseteq S^1 \cup \mathbb{R}_{>0}$.

Proof. According to Proposition 2.5.3, the characteristic polynomial of C is given by

$$\chi_C(X) = \prod_{i=1}^{p} [(X - \lambda_{i1})(X - \lambda_{i2})] \cdot (X-1)^{2(q-p)} \cdot (X+1)^{n-2q}.$$

Therefore we need to show that $\lambda_{ij} \subseteq S^1 \cup \mathbb{R}_{>0}$ for $i \in \{1, \ldots, p\}$, $j = 1, 2$. We make a case distinctions: In case $v_i \in (0, 4)$ we have $v_i(v_i - 4) < 0$ and therefore

$$\lambda_{ij} = \frac{1}{2} v_i - 1 \pm i \frac{1}{2} \sqrt{v_i(4 - v_i)}.$$

It follows

$$|\lambda_{ij}| = \left(\frac{1}{2} v_i - 1\right)^2 + \frac{1}{4} v_i(4 - v_i) = 1$$

and therefore $\lambda_{ij} \in S^1$. In case $v_i > 4$ we get $v_i(v_i - 4) > 0$ and therefore $\lambda_{ij} \in \mathbb{R}$. We also clearly have $\lambda_{i1} > \lambda_{i2}$. A straightforward computation shows that $\lambda_{i2} > 0$ and therefore $\lambda_{ij} \in \mathbb{R}_{>0}$. This finishes the proof. □

Second proof of Theorem 2.1.22 in the tree case. We already know by Proposition 2.2.8 that there is an eigenvalue λ of C such that $\lambda > 1$. Therefore $\rho > 1$ and since $\mathrm{spec}(C) \subseteq S^1 \cup \mathbb{R}_{>0}$ by Proposition 2.5.4 it follows that $\rho > 1$ is itself an eigenvalue of C and that all $\rho \neq \lambda \in \mathrm{spec}(C)$ satisfy $|\lambda| < \rho$. It remains to show that ρ is a simple eigenvalue of C. First of all, it is clear that $\rho = \lambda_{i1}$ for $i \in \{1, \ldots, p\}$ such that v_i is maximal among the eigenvalues of $E = M^t M$. Therefore it suffices to show that the biggest eigenvalue of E is simple. By the Perron-Frobenius Theorem 2.2.7 it suffices to show that E is irreducible. Therefore we want to better understand the entries of E:

Let $i, j \in \{1, \ldots, m\}$. Remember that M is just the lower left $\{m+1, \ldots, n\} \times \{1, \ldots, m\}$-block of the generalized Cartan matrix A. Therefore we have

$$E_{ij} = (M^t M)_{ij} = \sum_{k=1}^{n-m} (M^t)_{ik} M_{kj} = \sum_{k=m+1}^{n} A_{ki} A_{kj} = \sum_{k=m+1}^{n} \alpha_{ik} \alpha_{jk} \geq 0.$$

Now by definition of the graph underlying A, the property $E_{ij} > 0$ means that there is some $k \in \{m+1,\ldots,n\}$ such that both i and j are connected to it. This need not always be the case, but by assumption we know that A is connected. Therefore, and since vertices in $\{1,\ldots,m\}$ and vertices in $\{m+1,\ldots,n\}$ are not connected among each other, we get the following: for all $i, j \in \{1,\ldots,m\}$ there exist $i = i_1,\ldots,i_s = j \in \{1,\ldots,m\}$ and $k_1,\ldots,k_{s-1} \in \{m+1,\ldots,n\}$ such that for all $l \in \{1,\ldots,s-1\}$, k_l is connected to both i_l and i_{l+1}. The picture in $Q(A,\mathrm{id})$ looks as follows:

Therefore we get

$$\left(E^{s-1}\right)_{ij} = \sum_{i'_2,\ldots,i'_{s-1}=1}^{m} E_{i_1 i'_2} \cdots E_{i'_{s-1} i_s} \geq E_{i_1 i_2} \cdots E_{i_{s-1} i_s} > 0,$$

i.e. E is irreducible. This finished the proof. □

2.6 Coxeter transformations of the form $C(A,Q)$

So far, we have proved Theorem 2.1.22 in the case that $Q(A,\pi)$ is a tree. The remaining case to consider is the one where $Q(A,\pi)$ contains a cycle. As it turns out, a cycle in $Q(A,\pi)$ can never be oriented, and thus such a cycle has at least three vertices. Since for the investigations of this case the interplay with the quiver becomes even stronger, we start by giving a slightly more general definition of a Coxeter transformation, depending not on a permutation but on a quiver. We follow [Rin94] in this section, although we remark that the general definition of a Coxeter transformation was not explicitly stated in that paper.

Fix a generalized Cartan matrix $A \in \mathbb{R}^{I\times I}$, not necessarily symmetric. As before, $\alpha_{ij} = -A_{ij}$. Let Q be a *directed* quiver with vertex set $Q_0 = I$ (directed means that there is no oriented cycle of length ≥ 1 in Q). Let $\omega = i_0 \to \cdots \to i_m$ be a path in Q. Then we define $\alpha_\omega := \alpha_{i_0 i_1} \cdots \alpha_{i_{m-1} i_m}$. Note that for a path (i) of length zero at vertex $i \in I$ we get the empty product, so $\alpha_{(i)} = 1$. For $i \in Q_0 = I$ we define a vector $p(i) \in \mathbb{R}^I$ as follows: We set

$$p(i)_j := \sum_{\omega : i \rightsquigarrow j} \alpha_\omega, \quad j \in I,$$

where the sum runs over all paths ω from i to j in Q. Since Q is directed, the sum is finite, and so the definition makes sense.

Definition 2.6.1 (Coxeter transformation). We define the *Coxeter transformation* of A with respect to the quiver Q to be the linear transformation $C(A,Q) : \mathbb{R}^I \to \mathbb{R}^I$ satisfying

$$C(A,Q)\, e(x) := e(x) + \sum_{y\in I} \alpha_{xy} p(y).$$

We need further notation: For a vector $v \in \mathbb{R}^I$ and a subset $J \subseteq I$ we write $v|_J$ for the vector with entries $(v|_J)_i = v_i$ if $i \in J$ and $(v|_J)_i = 0$ else.

Now we can prove in the following Lemma and Proposition what we have already indicated before: That the earlier defined Coxeter transformation is just a special case of the Coxeter transformation with respect to a directed quiver:

Lemma 2.6.2. *Let $\pi : \{1, \dots, n\} \to I$ be a bijection. Then the quiver $Q = Q(A, \pi)$ is directed.*

Proof. For an arrow $\pi(i) \to \pi(j)$ we necessarily have $i < j$. Thus there cannot be any nontrivial oriented cycles. □

Proposition 2.6.3. *Let $\pi : \{1, \dots, n\} \to I$ be a bijection and $Q = Q(A, \pi)$. Let $L = R_{\pi(m)} \cdots R_{\pi(1)}$ as in Lemma 2.1.9. Then for all $m \in \{0, \dots, n\}$ we have:*

$$L(e(x)) = e(x) + \sum_{y \in I} \alpha_{xy} p(y)|_{\{\pi(1),\dots,\pi(m)\}}.$$

In particular, we have $C(A, \pi) = C(A, Q)$. This also means that two different bijections $\pi, \pi' : \{1, \dots, n\} \to I$ that lead to the same quiver $Q(A, \pi) = Q(A, \pi')$ give rise to the same Coxeter transformations.

Proof. We use Lemma 2.1.9:

$$\begin{aligned}
L(e(x)) &= e(x) + \sum_{i=1}^{m} \left[\sum_{M} \alpha_{x\pi(k_1)} \cdot \alpha_{\pi(k_1)\pi(k_2)} \cdots \alpha_{\pi(k_{|M|})\pi(i)} \right] e(\pi(i)) \\
&= e(x) + \sum_{i=1}^{m} \left[\sum_{\omega \text{ path, } t(\omega)=\pi(i)} \alpha_{xs(\omega)} \cdot \alpha_{\omega} \right] e(\pi(i)) \\
&= e(x) + \sum_{\omega \text{ path, } t(\omega)\in\{\pi(1),\dots,\pi(m)\}} \left(\alpha_{xs(\omega)} \alpha_{\omega} \right) e(t(\omega)) \\
&= e(x) + \sum_{y \in I} \alpha_{xy} p(y)|_{\{\pi(1),\dots,\pi(m)\}},
\end{aligned}$$

where the last equality is shown by looking at every entry of the vectors in $\mathbb{R}^I$ seperately. □

Therefore, we will forget about bijections altogether and just prove the remaining step of Theorem 2.1.22 for a directed quiver Q with certain properties:

Definition 2.6.4 (Quiver for A). A *quiver for A* is a directed quiver with vertex set I such that there is exactly one arrow between $x \neq y \in I$ if and only if $\alpha_{xy} \neq 0 \neq \alpha_{yx}$.

2.7 Obtaining a grip

In this section, we make a reduction to the case that we have a so-called *grip* in the quiver of our Coxeter transformation. As before, we follow [Rin94].

Remember that $(x_1, \dots, x_m)$ is called a source sequence in Q if x_i is a source in $\sigma_{x_{i-1}} \cdots \sigma_{x_1} Q$ for all i, where σ_{x_i} is the reflection on the vertex x_i.

Lemma 2.7.1. *Let Q directed and $(x_1, \dots, x_m)$ a source sequence in Q. Then $\sigma_{x_m} \cdots \sigma_{x_1} Q$ is still directed.*

Proof. By induction we only need to show that $\sigma_x Q$ is directed for any source $x \in Q_0$. Assume $\sigma_x Q$ is not directed. Then there is an oriented cycle $y_1 \to y_2 \to \cdots \to y_s \to y_1$ in $\sigma_x Q$ of length at least 1. If x was not one of the vertices $y_1, \dots, y_s$ then the same cycle would be an oriented cycle in Q as well, a contradiction. So one of the y_i equals x. But then x has an ingoing and an outgoing arrow in $\sigma_x Q$, contradicting the fact that x is a sink in $\sigma_x Q$. Thus $\sigma_x Q$ must be directed. □

We need a refinement of the concept of source sequences and admissible changes:

Definition 2.7.2 (Source sequence outside of J). Let $J \subseteq Q_0$. Then a source sequence $(x_1, \dots, x_s)$ in Q is called *source sequence outside of J* if $x_i \notin J$ for all i.

Definition 2.7.3 (Admissible change outside of J). Let $(x_1, \dots, x_s)$ be a source sequence outside of J. Then $\sigma_{x_s} \cdots \sigma_{x_1}$ is called an *admissible change of orientation outside of J*.

For the following two lemmas, we need further notation: Let $J \subset Q_0$ and set $J_0 := J$. Define inductively J_i as the set of vertices in Q_0 which belong to J_{i-1} or are a neighbour of a vertex in J_{i-1} (x is a neighbour of y if there is an arrow $x \to y$ or $y \to x$).

Lemma 2.7.4. *Let $J \subset Q_0$. Then a vertex $y \in J_m$ can occur at most m times in any given source sequence outside of J.*

Proof. The induction start $y \in J_0 = J$ is clear. Assume the statement is already proven for J_{m-1} and let $y \in J_m \setminus J_{m-1}$. Then y has a neighbour $z \in J_{m-1}$.

Let $(x_1, \dots, x_s)$ be a source sequence outside of J and let m' be the number of times y appears in it. Between every occurence of y while computing $\sigma_{x_s} \cdots \sigma_{x_1} Q$, every arrow ending in y has to be reversed at least once, since otherwise y would not be a source when it occurs the next time. That means that every neighbour of y occurs between every two occurences of y. Thus z occurs at least $m' - 1$ times in the source sequence. By induction, $m' - 1 \leq m - 1$, and so $m' \leq m$. □

Lemma 2.7.5. *Let Q be a connected directed quiver. Let $x \in Q_0$ and let $J = \{y \in Q_0 \mid x \rightsquigarrow y\}$ be the set of vertices that can be reached with a path starting at x. Then there is an admissible change of orientation ω outside of J such that x is the unique source in ωQ.*

Proof. If there is no source in $Q_0 \setminus \{x\}$, then we are done, since then x must be a (unique) source (remember that every directed quiver has a source). So assume that a source $y \in Q_0 \setminus \{x\}$ exists. Then $y \notin J$. We set $y_1 := y$. Assume $y_1, \dots, y_s$ are already defined and that there is a source y' in $\sigma_{y_s} \cdots \sigma_{y_1} Q$ unequal to x. Then we set $y_{s+1} := y'$. The resulting sequence $(y_1, \dots, y_{s+1})$ is then by induction a source sequence outside of J. We claim that this process has to stop:

Since Q is connected, we have $Q_0 = \bigcup_{m=0}^{r} J_m = J_r$ for some $r \in \mathbb{N}$. Then by Lemma 2.7.4 every element of Q_0 can occur at most r times in a source sequence outside of J. That means that source sequences outside of J have length at most $|Q_0| \cdot r$, which proofs the claim.

Let $(y_1, \dots, y_t)$ be a source sequence of maximal length constructed with the method above. Then by maximality, $\sigma_{y_t} \cdots \sigma_{y_1} Q$ cannot have a source different from x. But by Lemma 2.7.1, this quiver is still directed and thus must have a source, and so x is the unique source in it. □

Let from now on A be a (not necessarily symmetric) generalized Cartan matrix.

Lemma 2.7.6. *Assume Q is a connected quiver for A and contains a (not oriented) cycle of length at least three. Assume every vertex $z \in Q_0 = I$ in the cycle has the property $\sum_{x \in I} \alpha_{xz} < 1$. Then Q is of type $\tilde{\mathbb{A}}$, so it only consists of the cycle and additionally we have $\alpha_{xy} = 1$ for all $x \neq y$ with $\alpha_{xy} > 0$.*

Proof. Let z in the cycle. We have $\sum_{x \in I} \alpha_{xz} \leq 0$. Then $\sum_{x \in I \setminus \{z\}} \alpha_{xz} \leq 2$ and thus, as all the summands satisfy $\alpha_{xz} \geq 0$ we get that at most two of the α_{xz} are nontrivial. That means that z has at most two neighbours. Since z is in a cycle with at least three vertices, that means that z has exactly two neighbours and that for a neighbour x we have $\alpha_{xz} = 1$.

Therefore every element of the cycle has exactly two neighbours, namely the adjacent vertices in the cycle. As Q is connected that means that Q is this cycle. □

Definition 2.7.7. We say that A *is of type* $\tilde{\mathbb{A}}$ if – as in the conclusion of Lemma 2.7.6 – one quiver for A (and hence any quiver for A) is of type $\tilde{\mathbb{A}}$ and if for all $x \neq y$ with $\alpha_{xy} > 0$ we have $\alpha_{xy} = 1$.

Remark 2.7.8. If A is of type $\tilde{\mathbb{A}}$, then the graph with vertex set I and precisely α_{xy} edges between $x \neq y$ is itself of type $\tilde{\mathbb{A}}$.

The definition of a grip we choose is slightly less general then the one given in [Rin94], since we don't need it in full generality.

Definition 2.7.9 (Grip). Let Q be a quiver for A. A *grip* for (A, Q) is a path $i_0 \to i_1 \to \cdots \to i_t$ in Q such that the following properties are satisfied:

(i) i_0 is the only source of Q.

(ii) $\sum_{x \in I} \alpha_{x i_0} \geq 1$.

(iii) There is a path $i_0 = x_0 \to \cdots \to x_s = i_t$ with $x_1 \neq i_1$ and $x_{s-1} \neq i_{t-1}$.

(iv) For $0 < r < t$ there is only one path $i_0 \rightsquigarrow i_r$ and only one path $i_r \rightsquigarrow i_t$.

Remark 2.7.10. In a grip $i_0 \to \cdots \to i_t$ for (A, Q) and for $0 < r < t$, there is only one arrow ending in i_r, namely $i_{r-1} \to i_r$. Since if there is another $i_x \to i_r$, either i_x is a source or also the end of an arrow. Going on, we end up in the only source, i_0, contradicting the fact that $i_0 \to i_1 \to \cdots \to i_r$ is the only path $i_0 \rightsquigarrow i_r$.

Proposition 2.7.11. *Assume Q is a connected quiver for A and contains a cycle of length at least three. Then either A is of type $\tilde{\mathbb{A}}$ or there is an admissible change of orientation ω such that ωQ is a quiver for A and such that $(A, \omega Q)$ has a grip.*

Proof. Assume A is not of type $\tilde{\mathbb{A}}$. Then by Lemma 2.7.6 there is z in the cycle with $\sum_{x\in I}\alpha_{xz}\geq 1$. We label the cycle by $(i_0,\dots,i_{m-1})$ and assume $z=i_0$. Furthermore we extend the indices to $\mathbb{Z}$ by setting $i_r=i_{r'}$ in case $r\equiv r' \pmod m$. By Lemma 2.7.5 and 2.7.1 we can assume that i_0 is the unique source of Q.

Since Q is directed, there is a maximal u such that $i_0\to\cdots\to i_u$ is a path. Thus i_u is the endpoint of the two arrows $i_{u-1}\to i_u$ and $i_{u+1}\to i_u$. Take $1\leq t\leq u$ minimal with the property that at least two arrows end in i_t, $i_{t-1}\to i_t$ and $i_x\to i_t$ (We can not assume that x, which we fix from now on, lies on the cycle). Remember that $i_0=z$ was only chosen with the property that $\sum_{x\in I}\alpha_{xz}\geq 1$. We now further assume that z was chosen in such a way that $t=t(z)$ constructed here is minimal. Then we claim that $i_0\to\cdots\to i_t$ is a grip for (A,Q). By construction, properties (1) and (2) of Definition 2.7.9 are satisfied.

Since i_0 is the unique source and Q is directed and connected, there is a path $i_0\rightsquigarrow i_x\to i_t$. We write this second path as $i_0=x_0\to x_1\to\cdots\to x_s=i_t$) and want to show that $x_1\neq i_1$ and $x_{s-1}\neq i_{t-1}$ (which then proves property (3)): $x_{s-1}=x\neq i_{t-1}$ is clear by definition of t. Assume $x_1=i_1$. Let k be the maximal index with $x_k=i_k$. Then $i_k=x_k\to x_{k+1}\neq i_{k+1}$ proves that i_k has two outgoing and one ingoing arrow. But then i_k has by construction the desired property $\sum_{y\in I}\alpha_{yi_k}\geq 1$, which allows it to set $i_0=i_k$ in the beginning of this proof. After an admissible change of orientation as in Lemma 2.7.5, i_k would be the unique source. But this admissible change does not affect vertices that were originally reached with a path starting in i_k, which means that after the change, $i_k\to i_{k+1}\to\cdots\to i_t$ and $i_k=x_k\to x_{k+1}\to\cdots\to x_{s-1}=x\to i_t$ are still paths. But then i_t has still two ingoing arrows, which violates the minimality of $t=t(z)$. That shows that $x_1\neq i_1$, as desired.

Now we proof property (4): Let $0<r<t$. Since $i_1,\dots,i_r$ all have by construction only one ingoing arrow, there is clearly only one path $i_0\rightsquigarrow i_r$. Now we show that there is also only one path $i_r\rightsquigarrow i_t$: Assume that there is another path $i_r=y_0\to y_1\to\cdots\to y_k=i_t$ and assume that r is maximal allowing this. Then $y_1\neq i_{r+1}$. This means that i_r has at least three neighbours. By the same argument as before, we can make i_r to the unique source without affecting the two paths $i_r\rightsquigarrow i_t$, and thus we get a contradiction to the minimality of t again. This finishes the proof. □

Lemma 2.7.12. *Let A be of type $\tilde{\mathbb{A}}$. Then A is positive semidefinite, i.e. the associated quadratic form q_A is positive semidefinite.*

Proof. This is somehow clear since $\tilde{\mathbb{A}}$ is a Euclidean diagram and since the quadratic form of a Euclidean quiver is positive semidefinite. But we can do the proof also directly:

Order the set I in such a way that the cycle of the graph of A is of the form $1-2-\cdots-n-1$. Consider $I=\{1,\dots,n\}$ as the underlying set of the group $\mathbb{Z}/n\mathbb{Z}$. Then we have $\alpha_{ii}=2$, $\alpha_{ij}=1$ whenever $i=j\pm 1$ and $\alpha_{ij}=0$ else. We get

$$q_A(x)=x^tAx=\sum_{i,j=1}^n x_iA_{ij}x_j=\sum_{i=1}^n 2x_i^2-2\sum_{i>j}x_i\alpha_{ij}x_j=\sum_{i=1}^n 2x_i^2-2\sum_{i=1}^n x_ix_{i-1}$$
$$=\sum_{i=1}^n\left(x_i^2-2x_ix_{i-1}+x_{i-1}^2\right)=\sum_{i=1}^n(x_i-x_{i-1})^2\geq 0,$$

so A is indeed positive semidefinite. □

We finish this section by giving a reduction for proving Theorem 2.1.22:

Proposition 2.7.13. *Let A be a connected, indefinite, symmetric, generalized Cartan matrix. Assume that for every quiver Q for A such that (A, Q) has a grip, we could prove the conclusions of Theorem 2.1.22, i.e.:*

(i) Let ρ be the spectral radius of $C = C(A, Q)$. Then $\rho > 1$ is itself an eigenvalue of C with algebraic multiplicity one.

(ii) If $\rho \neq \lambda \in \operatorname{spec}(C)$, then $|\lambda| < \rho$.

Then Theorem 2.1.22 would be proven completely.

Proof. Let $\pi : \{1, \ldots, n\} \to I$ be a bijection. Then according to section 2.4 or 2.5, Theorem 2.1.22 is proven in the case that $Q = Q(A, \pi)$ is a tree. Thus we can assume that there is a cycle in Q. Since there are no double arrows between vertices and since Q is directed, this cycle has length at least three. Furthermore, A is not of type $\tilde{\mathbb{A}}$ according to Lemma 2.7.12 since A is indefinite. Therefore, by Proposition 2.7.11 there is an admissible change of orientation ω such that $(A, \omega Q)$ has a grip. The Coxeter transformations $C(A, \pi)$ and $C(A, \omega Q)$ are conjugate to each other by Lemma 2.3.7 and Propositions 2.3.8 and 2.6.3. Therefore they have the same spectral properties and we are done if we showed the theorem for $C(A, \omega Q)$. □

2.8 An invariant cone

Fix as usual a generalized (not necessarily symmetric) Cartan matrix A. We further fix a quiver Q for A such that (A, Q) has a grip, which makes sense due to Proposition 2.7.13. we write the grip as $0 \to 1 \to \cdots \to t$ and define $G = \{0, 1, \ldots, t\} \subseteq I = Q_0$. As before, we follow [Rin94], but we mention that the formulas in Lemma 2.8.6 were not stated explicitly in that paper and make computations more transparent.

For $i \notin G$ we define $b(i) = e(i)$, and for $i \in G$ we set $b(i) = \sum_{j=i}^{t} e(j)$. They form a basis for $\mathbb{R}^I$ (For example, $e(i) = b(i) - b(i+1)$ for $i \in \{0, \ldots, t-1\}$, so the $b(i)$ generate $\mathbb{R}^I$).

Definition 2.8.1 (Cone for (A, Q)). We define the *cone $\mathcal{K}$ for (A, Q)* to be the set generated by non-negative linear combinations of the $b(i)$, that is

$$\mathcal{K} = \left\{ \sum_{i \in I} \lambda_i b(i) \mid \lambda_i \geq 0 \right\}.$$

We will show that $\mathcal{K}$ is invariant under the Coxeter transformation $C = C(A, Q)$ and that some positive power of C even sends $\mathcal{K} \setminus \{0\}$ into the interior $\mathring{\mathcal{K}}$. Then some version of the Perron-Frobenius theorem will prove Theorem 2.1.22.

Lemma 2.8.2. *The cone $\mathcal{K}$ is a closed subset of $\mathbb{R}^I$. Moreover, the interior of $\mathring{\mathcal{K}}$ is given by the strictly positive linear combinations of the $b(i)$:*

$$\mathring{\mathcal{K}} = \left\{ \sum_{i \in I} \lambda_i b(i) \mid \lambda_i > 0 \right\}.$$

Proof. Both claims are true since the $b(i)$ form a basis: There is a linear isomorphism $\mathbb{R}^I \to \mathbb{R}^I$ with $b(i) \mapsto e(i)$, and linear isomorphisms on finite-dimensional vector spaces are homeomorphisms. Then just observe that the statement is clearly true when we replace $b(i)$ by $e(i)$. □

Lemma 2.8.3. *We can describe $\mathcal{K}$ and $\mathring{\mathcal{K}}$ alternatively as follows:*

$$\mathcal{K} = \{c \in \mathbb{R}^I \mid c_i \geq 0,\ c_0 \leq \cdots \leq c_t\},$$
$$\mathring{\mathcal{K}} = \{c \in \mathbb{R}^I \mid c_i > 0,\ c_0 < \cdots < c_t\}.$$

Proof. Let $c \in \mathcal{K}$. Write $c = \sum_{j \in I} \lambda_j b(j)$ with $\lambda_j \geq 0$. Since the $b(j)$ have all non-negative entries, the same follows for c. For $0 \leq i \leq t$ we have

$$c_i = \sum_{j \in I} \lambda_j b(j)_i = \sum_{j=0}^{t} \lambda_j \sum_{k=j}^{t} e(k)_i = \sum_{j=0}^{i} \lambda_j \cdot 1 = \sum_{j=0}^{i} \lambda_j,$$

so we clearly get $c_0 \leq c_1 \leq \cdots \leq c_t$.

On the other hand, if $c \in \mathbb{R}^I$ has these properties, then we can define $\lambda_j = c_j$ for $j \notin G$ and $\lambda_j = c_j - c_{j-1}$ for $0 \leq j \leq t$ (where $c_{-1} := 0$) and thus get a vector $\sum_{j \in I} \lambda_j b(j) \in \mathcal{K}$ which equals c. Basically the same proof works for $\mathring{\mathcal{K}}$. □

In the following, we set $m(I) := \sum_{i \in I} e(i)$. Also remember that $p(i)$ was defined as $p(i)_j = \sum_{\omega : i \rightsquigarrow j} \alpha_\omega$.

Lemma 2.8.4. *If $i \neq j$, then $p(i)_j = \sum_{j' \to j} p(i)_{j'} \alpha_{j'j} = \sum_{i \to i'} \alpha_{ii'} p(i')_j$. In particular we have $p(i)_{j'} \leq p(i)_j$ respectively $p(i')_j \leq p(i)_j$ when there are arrows $j' \to j$ respectively $i \to i'$ (this holds clearly even when $i = j$).*

Proof.

$$\begin{aligned}\sum_{j' \to j} p(i)_{j'} \alpha_{j'j} &= \sum_{j' \to j} \sum_{\omega' : i \rightsquigarrow j'} \alpha_{\omega'} \alpha_{j'j} \\ &= \sum_{\omega : i \to j} \alpha_\omega = p(i)_j,\end{aligned}$$

which simply follows by noticing that $\alpha_{\omega'} \alpha_{j'j} = \alpha_\omega$ for the concatenation $\omega : i \rightsquigarrow j' \to j$ of ω' and the unique arrow $j' \to j$ and that on the other hand, every α_ω splits in this way since $i \neq j$. The other equality is similar. □

Lemma 2.8.5. *We have $m(I) \in \mathcal{K}$ and $p(i) \in \mathcal{K}$ for all $i \in I$.*

Proof. We have $m(I) = b(0) + \sum_{i \notin G} b(i) \in \mathcal{K}$.

The vector $p(i)$ is entrywise non-negative since every α_ω for a path ω is non-negative (observe that factors $\alpha_{ii} = -2 < 0$ never occur, since for example $\alpha_{(i)} = 1$ is the empty product). Furthermore, if there is an arrow $j' \to j$, then by Lemma 2.8.4 we get $p(i)_{j'} \leq p(i)_j$. So we get $p(i)_0 \leq p(i)_1 \leq \cdots \leq p(i)_t$ and thus $p(i) \in \mathcal{K}$ by Lemma 2.8.3. □

We need some further notation: For $x \in I$ we set $[x] := \{x\}$ if $x \notin G$ and $[x] := \{x, x+1, \ldots, t\}$ in case $x \in G$. Furthermore we extend the usual definition of the Kronecker delta by setting $\delta_{[x]j} = 1$ in case $j \in [x]$ and 0 else. Clearly, $\delta_{[x]j} = \sum_{i \in [x]} \delta_{ij}$. In addition we set $\alpha_{[x]j} = \sum_{i \in [x]} \alpha_{ij}$. We aim to learn how the different entries of images under the Coxeter transformation relate to each other:

Lemma 2.8.6. *Let $x \in I$ and set $c := Cb(x)$. Then the j-th entry of c equals*

$$c_j = \delta_{[x]j} + \alpha_{[x]j} + \sum_{j' \to j} \left(c_{j'} - \delta_{[x]j'}\right) \alpha_{j'j}.$$

More generally, if $v \in \mathbb{R}^I$ is any vector and we define $c := Cv$, then the j-th entry if c equals

$$c_j = -v_j + \sum_{j \to j'} v_{j'} \alpha_{j'j} + \sum_{j' \to j} c_{j'} \alpha_{j'j}.$$

Proof. Set $d := Ce(x) - e(x) = \sum_{i \in I} \alpha_{xi} p(i)$. Then using Lemma 2.8.4 we get

$$\begin{aligned} d_j &= \sum_{i \in I} \alpha_{xi} p(i)_j = \alpha_{xj} + \sum_{i \in I \setminus \{j\}} \alpha_{xi} p(i)_j = \alpha_{xj} + \sum_{i \in I \setminus \{j\}} \alpha_{xi} \sum_{j' \to j} p(i)_{j'} \alpha_{j'j} \\ &= \alpha_{xj} + \sum_{j' \to j} \left(\sum_{i \in I \setminus \{j\}} \alpha_{xi} p(i)_{j'} \right) \alpha_{j'j} = \alpha_{xj} + \sum_{j' \to j} d_{j'} \alpha_{j'j}, \end{aligned}$$

where in the last equality we used that $p(j)_{j'} = 0$ since there is no oriented cycle in Q. From this it follows that

$$(Ce(x))_j = \delta_{xj} + d_j = \delta_{xj} + \alpha_{xj} + \sum_{j' \to j} \left((Ce(x))_{j'} - \delta_{xj'} \right) \alpha_{j'j}.$$

The first result follows by summing up this formular for all $i \in [x]$ in place of x. For the second formula, we compute

$$\begin{aligned} c_j &= \sum_{x \in I} v_x (Ce(x))_j \\ &= \sum_{x \in I} v_x \left[\delta_{xj} + \alpha_{xj} + \sum_{j' \to j} \left((Ce(x))_{j'} - \delta_{xj'} \right) \alpha_{j'j} \right] \\ &= v_j + \sum_{x \in I} v_x \alpha_{xj} + \sum_{j' \to j} c_{j'} \alpha_{j'j} - \sum_{x \in I} \sum_{j' \to j} \delta_{xj'} v_x \alpha_{j'j} \\ &= v_j + \sum_{x \in I} v_x \alpha_{xj} + \sum_{j' \to j} c_{j'} \alpha_{j'j} - \sum_{x \to j} v_x \alpha_{xj} \\ &= v_j - 2v_j + \sum_{j \to x} v_x \alpha_{xj} + \sum_{j' \to j} c_{j'} \alpha_{j'j} \\ &= -v_j + \sum_{j \to j'} v_{j'} \alpha_{j'j} + \sum_{j' \to j} c_{j'} \alpha_{j'j}, \end{aligned}$$

where in the second to last step we used that $\alpha_{xj} = 0$ except in case that $x \to j$, $j \to x$ or $x = j$. □

Lemma 2.8.7. *Let x be not a source in Q. Then $Ce(x)$ is entrywise non-negative.*

Proof. According to Lemma 2.8.6 we have

$$(Ce(x))_0 = \alpha_{x0} \geq 0.$$

Now let $x \neq j \neq 0$ such that there is a path $j \rightsquigarrow x$. Then

$$(Ce(x))_j = \alpha_{xj} + \sum_{j' \to j} \left((Ce(x))_{j'} - \delta_{xj'}\right) \alpha_{j'j}. \tag{2.6}$$

By induction, we have $(Ce(x))_{j'} \geq 0$ for all the appearing j'. Furthermore, $j' = x$ can not occur in the right sum, since otherwise we would have an oriented cycle. Therefore we get $(Ce(x))_j \geq 0$. In case that there is even an arrow $j \to x$ we see that $(Ce(x))_j > 0$.

Now consider the case $j = x$. Then we have

$$(Ce(x))_x = -1 + \sum_{j' \to x} (Ce(x))_{j'} \alpha_{j'x}.$$

Since x is no source, there is at least one arrow $j' \to x$. From before we know that $(Ce(x))_{j'} \geq 1$ and so we conclude $(Ce(x))_x \geq 0$. Now in the remaining case that $x \neq j \neq 0$ and that there is no path $j \rightsquigarrow x$ we get again formula 2.6. By induction we can assume that all the appearing $(Ce(x))_{j'}$ are non-negative. It could now happen that $j' = x$ is one of the indices in the sum, but then the negative term $-\alpha_{xj}$ gets swallowed from α_{xj} on the left and we again get $(Ce(x))_j \geq 0$. This finishes the proof. □

In the following three lemmas we show that C maps the cone $\mathcal{K}$ to itself.

Lemma 2.8.8. *If $x \notin G$, then $Cb(x) \in \mathcal{K}$.*

Proof. We have $Cb(x) = Ce(x) = c$ in the notation of Lemma 2.8.6. Then since x is not a source ($0 \in G$ is the only source) we see by Lemma 2.8.7 that c is entrywise non-negative. So by Lemma 2.8.3 we only need to show that $c_0 \leq c_1 \leq \cdots \leq c_t$. For $j \in \{1, \ldots, t-1\}$ we get by Lemma 2.8.6, using that $j-1 \to j$ is the only arrow ending in j,

$$c_j = \delta_{xj} + \alpha_{xj} + \left(c_{j-1} - \delta_{x,j-1}\right) \alpha_{j-1,j} = \alpha_{xj} + c_{j-1}\alpha_{j-1,j} \geq c_{j-1}.$$

For $j = t$ we have

$$c_t = \delta_{xt} + \alpha_{xt} + \sum_{j' \to t} \left(c_{j'} - \delta_{xj'}\right) \alpha_{j't} = \alpha_{xt} + c_{t-1}\alpha_{t-1,t} + \sum_{j' \neq t-1,\ j' \to t} \left(c_{j'} - \delta_{xj'}\right) \alpha_{j't}.$$

In case there is no arrow $x \to t$ we have $\delta_{xj'} = 0$ for all j' in the right sum. Together with the fact that all $c_{j'}$ are non-negative we get $c_t \geq c_{t-1}$. In case there is an arrow $x \to t$ we get

$$c_t = \alpha_{xt} + c_{t-1}\alpha_{t-1,t} + (c_x - 1)\,\alpha_{xt} + \sum_{j' \notin \{t-1,x\},\ j' \to t} c_{j'}\alpha_{j't} \geq c_{t-1},$$

finishing the proof. □

Lemma 2.8.9. *Let $0 \leq i \leq t$ and $c := Cb(i)$. Then we have*

(i) $c_0 \geq 0$. *In case $i = 1$ we even have $c_0 \geq 1$.*

(ii) Let $x \notin \{1, \dots, t\}$ such that there is a path $x \rightsquigarrow t$. Then $c_x \geq 0$.

(iii) Let $x \neq t-1$ such that there is an arrow $x \to t$. Then we even have $c_x \geq 1$.

Proof. For (i) we just observe

$$c_0 = \delta_{[i]0} + \alpha_{[i]0} \geq 0,$$

which is best proven by doing a case distinction between $i = 0$ and $i > 0$. In case $i = 1$ we even see $c_0 \geq 1$, due to the summand $\alpha_{10} \geq 1$.

Next we prove (ii). We do it by induction on the length of the longest path $0 \rightsquigarrow x$, the induction start being done in (i), i.e. in case $x = 0$. So assume $x \notin G$:

$$c_x = \alpha_{[i]x} + \sum_{j' \to x} \left(c_{j'} - \delta_{[i]j'}\right) \alpha_{j'x}.$$

If $j' \neq 0$ in the right sum then $j' \notin G$ since there is a path $x \rightsquigarrow t$ and G is a grip. Then by induction $c_{j'} \geq 0$. If $j' = 0$ in the sum, then the summand $\left(c_0 - \delta_{[i]0}\right) \alpha_{0x}$ occurs. In the bad case $i = 0$, the summand $-\alpha_{0x}$ emerging from this gets swallowed by the sum $\alpha_{[0]x}$. All in all we see $c_x \geq 0$.

Now we prove (iii): By Lemma 2.8.6 we have

$$c_x = \delta_{[i]x} + \alpha_{[i]x} + \sum_{j' \to x} \left(c_{j'} - \delta_{[i]j'}\right) \alpha_{j'x}.$$

In case $x = 0$ this transforms to

$$c_0 = \delta_{[i]0} + \alpha_{[i]0} = \delta_{[i]0} + \alpha_{t0} + \sum_{k=i}^{t-1} \alpha_{k0},$$

which is $-1 + \alpha_{t0} + \alpha_{10} + \sum_{k=2}^{t-1} \alpha_{k0} \geq 1$ in case $i = 0$ (note that $0 = x \neq t-1$ and so $1 \leq t-1$ occurs as index in the sum) and $\alpha_{t0} + \sum_{k=i}^{t-1} \alpha_{k0} \geq 1$ in case $i \neq 0$.

In case $x \neq 0$ we get

$$c_x = \alpha_{[i]x} + \sum_{j' \to x} \left(c_{j'} - \delta_{[i]j'}\right) \alpha_{j'x} = \alpha_{tx} + \sum_{k=i}^{t-1} \alpha_{kx} + \sum_{j' \notin G, j' \to x} c_{j'} \alpha_{j'x} + \left(c_0 - \delta_{[i]0}\right) \alpha_{0x},$$

where the last summand is only present if there is an arrow $0 \to x$. We know by (ii) that all the appearing $c_{j'}$ are non-negative. A summand $-a_{0x}$ only occurs if $i = 0$ in which case it gets swallowed by the sum $\sum_{k=0}^{t-1} \alpha_{kx}$. All in all we see $c_x \geq \alpha_{tx} \geq 1$. □

Lemma 2.8.10. *Let $0 \leq i \leq t$ and $c := Cb(i)$. Then $c \in \mathcal{K}$.*

Proof. We first show $0 \leq c_0 \leq \cdots \leq c_{t-1}$. We view i as fixed and make case distinctions in j (where some cases can be empty): For $1 \leq j \leq i-1$ we have by Lemma 2.8.6

$$c_j = \alpha_{[i]j} + c_{j-1}\alpha_{j-1,j} \geq c_{j-1},$$

provided $c_{j-1} \geq 0$. By induction we are done since $c_0 \geq 0$ by Lemma 2.8.9. Note for the upcoming lemma that in case $j = i-1$ and $i \geq 2$ we even have a strict inequality since $c_{i-1} = \alpha_{[i]i-1} + c_{i-2}\alpha_{i-2,i-1} \geq \alpha_{i,i-1} + c_{i-2} \geq 1 + c_{i-2}$. Next we look at the case $i \leq j \leq t-1$ and $j \geq 1$. Then we have

$$c_j = 1 + \alpha_{[i]j} + \left(c_{j-1} - \delta_{[i]j-1}\right)\alpha_{j-1,j}.$$

In case $j = i$ we get

$$c_j = -1 + \alpha_{[j+1]j} + c_{j-1}\alpha_{j-1,j} = \left(-1 + \alpha_{j+1,j}\right) + \alpha_{[j+2]j} + c_{j-1}\alpha_{j-1,j} \geq c_{j-1},$$

and in case $j > i$ we get

$$\begin{aligned} c_j &= -1 + \left(\sum_{k \notin \{j-1,j\}, i \leq k \leq t} \alpha_{kj}\right) + \alpha_{j-1,j} + c_{j-1}\alpha_{j-1,j} - \alpha_{j-1,j} \\ &= \left(-1 + \alpha_{j+1,j}\right) + \left(\sum_{k \notin \{j-1,j,j+1\}, i \leq k \leq t} \alpha_{kj}\right) + c_{j-1}\alpha_{j-1,j} \geq c_{j-1}. \end{aligned}$$

All in all we have shown $0 \leq c_0 \leq \cdots \leq c_{t-1}$.

Now we show $c_t \geq c_{t-1}$:

$$\begin{aligned} c_t &= \delta_{[i]t} + \alpha_{[i]t} + \sum_{j' \to t} \left(c_{j'} - \delta_{[i]j'}\right)\alpha_{j't} \\ &= -1 + \sum_{k=i}^{t-1} \alpha_{kt} + \left(c_{t-1} - \delta_{[i],t-1}\right)\alpha_{t-1,t} + \sum_{j' \neq t-1, j' \to t} \left(c_{j'} - \delta_{[i]j'}\right)\alpha_{j't}. \end{aligned}$$

In case $i = t$ we get

$$c_t = -1 + c_{t-1}\alpha_{t-1,t} + \sum_{j' \neq t-1, j' \to t} c_{j'}\alpha_{j't} \geq c_{t-1},$$

since by definition of a grip we know that there is an additional arrow $j' \to t$ with $j' \neq t-1$ and since $c_{j'} \geq 1$ for these arrows by Lemma 2.8.9. In case $i \neq t$ we get

$$\begin{aligned} c_t &= -1 + \sum_{k=i}^{t-2} \alpha_{kt} + \alpha_{t-1,t} + c_{t-1}\alpha_{t-1,t} - \alpha_{t-1,t} + \sum_{j' \neq t-1, j' \to t} \left(c_{j'} - \delta_{[i]j'}\right)\alpha_{j't} \\ &= -1 + \sum_{k=i}^{t-2} \alpha_{kt} + c_{t-1}\alpha_{t-1,t} + \sum_{j' \neq t-1, j' \to t} \left(c_{j'} - \delta_{[i]j'}\right)\alpha_{j't}. \end{aligned}$$

This could only be smaller than c_{t-1} if the right sum is zero, which can only happen if there is an arrow $0 \to t$ and if there is no other arrow $j' \to t$ with $j' \neq t-1$. Additionally, we would have $\sum_{k=i}^{t-2} \alpha_{kt} = 0$, which can only be true if $i > 0$ since $\alpha_{0t} > 0$. But then $\delta_{[i]0} = 0$, so $\left(c_0 - \delta_{[i]0}\right)\alpha_{0t} > 0$, thus we indeed get $c_t \geq c_{t-1}$.

We end by showing that $c_x \geq 0$ for all $x \in I$ (so far we didn't show it for those $x \notin G$ that do not allow a path $x \rightsquigarrow t$). We do it by induction on the length of the longest path from any element in G to x. The induction start is already done, since this is the case $x \in G$. Then for $x \notin G$ we get

$$\begin{aligned} c_x &= \alpha_{[i]x} + \sum_{j' \to x} \left(c_{j'} - \delta_{[i]j'}\right)\alpha_{j'x} \\ &= \alpha_{[i]x} - \sum_{i \leq j' \leq t,\ j' \to x} \alpha_{j'x} + \sum_{j' \to x} c_{j'}\alpha_{j'x} \\ &\geq \sum_{j' \to x} c_{j'}\alpha_{j'x} \geq 0, \end{aligned}$$

where we used the induction hypotheses in the last step. □

Corollary 2.8.11. *C maps $\mathcal{K}$ into itself.*

2.9 A strongly invariant cone

Let the notations and conventions as in the preceding section. We still follow [Rin94].

In the following lemmata we aim to show that some positive power of C even maps $\mathcal{K} \setminus \{0\}$ into $\mathring{\mathcal{K}}$. For this we need new notation: for vectors $c, d \in \mathbb{R}^I$, we write $c \preceq d$ in case $d - c \in \mathcal{K}$. This is a partial order on $\mathbb{R}^I$, i.e. it is reflexive, transitive and antisymmetric.

Lemma 2.9.1. *We have the following:*

(i) *For $1 \leq i \leq t$ it is $b(i-1) \preceq Cb(i)$.*

(ii) *It is $b(0) \preceq Cb(0)$ if and only if $\alpha_{10} \geq 2$ or $0 \to t$.*

(iii) *In case $b(0) \npreceq Cb(0)$ there is an arrow $x \to t$ with $b(x) \preceq Cb(0)$. In fact, every arrow $x \to t$ with $x \neq t-1$ has this property.*

Proof. Note that for any vector $c \in \mathcal{K}$ we have $b(0) \preceq c$ if and only if $c - b(0) \in \mathcal{K}$, which is the case if and only if $c_0 \geq 1$ (since we already have $c_0 - 1 \leq \cdots \leq c_t - 1$ and since the other indices are not effected by subtracting $b(0)$ from c). We further know from Lemma 2.8.9 (i) that $(Cb(1))_0 \geq 1$. This together with Lemma 2.8.10 shows that $b(0) \preceq Cb(1)$. Now assume that $i \geq 2$. We want to show $b(i-1) \preceq Cb(i)$, which is by similar considerations equivalent to saying that $(Cb(i))_{i-2} + 1 \leq (Cb(i))_{i-1}$. But this fact was already noted in the proof of Lemma 2.8.10, which finishes the proof of (i).

For (ii) we use again that $b(0) \preceq Cb(0)$ if and only if $(Cb(0))_0 \geq 1$, i.e. $\delta_{[0]0} + \alpha_{[0]0} = -1 + \alpha_{10} + \alpha_{t0} \geq 1$. This is equivalent to $\alpha_{10} \geq 2$ or $\alpha_{t0} \geq 1$, which means there is an arrow $0 \to t$, proving (ii).

For (iii) assume $b(0) \npreceq Cb(0)$. Then $\alpha_{10} + \alpha_{t0} \leq 1$, so $\alpha_{t0} = 0$. We know by definition of a grip that there has to be an arrow $x \to t$ with $x \neq t-1$ and since x cannot be 0 since $\alpha_{t0} = 0$, we know by the grip properties that $x \notin G$. Then $b(x) = e(x)$. Furthermore, $(Cb(0))_x \geq 1$ by Lemma 2.8.9 (iii). Then $Cb(0) - b(x)$ is entrywise non-negative, and thus since $Cb(0) \in \mathcal{K}$ we get $b(x) \preceq Cb(0)$, proving (iii). □

Remember that $m(I) = \sum_{i \in I} e(i)$.

Lemma 2.9.2. *We have* $m(I) + e(t) \leq Cm(I)$.

Proof. Set $c := Cm(I)$. Observe that the first formula from Lemma 2.8.6 also holds for I instead of $[x]$. So we get

$$c_j = 1 + \alpha_{Ij} + \sum_{j' \to j} (c_{j'} - 1)\,\alpha_{j'j}$$

for all $j \in I$. That means

$$c_0 = 1 + \sum_{x \in I} \alpha_{x0} \geq 2,$$

since G is a grip with source 0. Now we show that for $j \in I$ not a sink we have $c_j \geq c_0$. We show that by induction on the longest path $0 \rightsquigarrow j$. We have in general

$$c_j = -1 + \sum_{j' \neq j} \alpha_{j'j} + \sum_{j' \to j} c_{j'}\alpha_{j'j} - \sum_{j' \to j} \alpha_{j'j} = -1 + \sum_{j' \to j} c_{j'}\alpha_{j'j} + \sum_{j \to j'} \alpha_{j'j},$$

since whenever there is no arrow between j and j' we have $\alpha_{j'j} = 0$. Since j is not a sink, the last summand is greater or equal to 1, so by induction we get $c_j \geq -1 + \sum_{j' \to j} c_0 \alpha_{j'j} + 1 \geq c_0$, since c_j is also not a source unless $j = 0$.

Let j be a sink. Then j is not a source, so there is an arrow $j' \to j$, so $c_j \geq -1 + c_{j'} \geq -1 + c_0 \geq -1 + 2 = 1$ (Here we used that $c = Cb(0) + \sum_{x \notin G} Cb(x) \in \mathcal{K}$, so this vector is entrywise non-negative). All in all this shows that $c_j \geq 1$ for all $j \in I$, proving that $m(I) \leq c$. In order to see that even $m(I) + e(t) \leq c$, we need by similar considerations as before only check that $c_{t-1} + 1 \leq c_t$. We know that there is an arrow $x \to t$ with $x \neq t - 1$. Therefore, since all $c_{j'}$ are greater than or equal to 1 we get

$$\begin{aligned} c_t &= 1 + \alpha_{It} + \sum_{j' \to t} (c_{j'} - 1)\,\alpha_{j't} \\ &\geq -1 + \alpha_{t-1,t} + \alpha_{x,t} + (c_{t-1} - 1)\,\alpha_{t-1,t} + (c_x - 1)\,\alpha_{xt} \\ &\geq -1 + c_{t-1} + c_x \geq c_{t-1} + c_0 - 1 \geq c_{t-1} + 1, \end{aligned}$$

since x is not a sink. This finishes the proof. □

Definition 2.9.3 (Property P(i)). Let $v \in \mathbb{R}^I$ and $i \in I$. Then we say that v *has property* $P(i)$ if $v \in \mathcal{K}$, $v_i \geq 1$ and $v_i \geq v_j + 1$ for all $j \to i$.

Lemma 2.9.4. *Let* $v \in R^I$ *have property* $P(i)$. *Then* $c := Cv$ *has property* $P(j)$ *for all* $j \to i$.

Proof. Let $j \to i$. Then by the second formula in Lemma 2.8.6 we have

$$c_j = -v_j + \sum_{j \to j'} v_{j'}\alpha_{j'j} + \sum_{j' \to j} c_{j'}\alpha_{j'j} \geq -v_j + v_i\alpha_{ij} + \sum_{j' \to j} c_{j'} \geq 1 + \sum_{j' \to j} c_{j'},$$

where we used that $v_{j'} \geq 0$ for all j' (since $v \in \mathcal{K}$) and that $v_i \geq v_j + 1$. Now since $c \in \mathcal{K}$ by Corollary 2.8.11, we have $c_{j'} \geq 0$ for all appearing j'. So we get $c_j \geq 1$ and $c_j \geq c_{j'} + 1$ for all $j' \to j$. □

Lemma 2.9.5. *Let $v \in \mathbb{R}^I$ with property $P(i)$ and let $j \rightsquigarrow i$. Let t_i be the length of some path from j to i. Then $C^{t_i} v$ has property $P(j)$.*

Proof. We just use induction on $t(i)$ and Lemma 2.9.4. □

Lemma 2.9.6. *Let $v \preceq w$ for $v, w \in \mathcal{K}$. Then $Cv \preceq Cw$.*

Proof. By Corollary 2.8.11 we have $Cw - Cv = C(w - v) \in \mathcal{K}$ since $w - v \in \mathcal{K}$. □

Lemma 2.9.7. *Let $1 \neq y \in I$ be a neighbour of 0, i.e. there is an arrow $0 \to y$. Assume further that $\alpha_{10} = 1$ and $\alpha_{t0} = 0$. Then for any $i \in I$ there is $0 \leq t_i \leq h$ such that $b(y) \preceq C^{t_i} b(i)$, where h is the length of the longest path in Q.*

Proof. The relation is equivalent to $(C^{t_i} b(i))_y \geq 1$. We make a case distinction:

In the first case assume $i \neq 0$. Let t_i be the length of some path $0 \rightsquigarrow i$. Then $t_i \geq 1$. Set $v := C^{t_i - 1} b(i)$. We aim to show that $(Cv)_y \geq 1$. By the second formula in Lemma 2.8.6 we get

$$\begin{aligned} (Cv)_y &= -v_y + \sum_{y \to j'} v_{j'} \alpha_{j'y} + \sum_{j' \to y} (Cv)_{j'} \alpha_{j'y} \\ &\geq -v_y + (Cv)_0 \geq -v_y + 1, \end{aligned}$$

where we used Lemma 2.9.5 in the last inequality. Now if $v_y = 0$ we are done. If $v_y \geq 1$ on the other hand, then we already have $\left(C^{t_i - 1} b(i)\right)_y \geq 1$ and are thus already done with a smaller exponent.

Consider the remaining case $i = 0$. According to Lemma 2.9.1 (ii) and (iii) there is an arrow $x \to t$ such that $b(x) \preceq Cb(0)$. We have $x \neq 0$ since 0 is not a neighbour of t. Let l_x be the length of a path $0 \rightsquigarrow x$. By the first case we get $\left(C^{l_x} b(x)\right)_y \geq 1$ (or possibly $\left(C^{l_x - 1} b(x)\right)_y \geq 1$) and so $b(y) \preceq C^{l_x} b(x) \preceq C^{l_x + 1} b(0)$ (or $b(y) \preceq C^{l_x - 1} b(x) \preceq C^{l_x} b(0)$), where we have $l_x + 1 \leq h$ since x is no sink. □

Lemma 2.9.8. *For all $i \in I$ we have $m(I) \preceq C^{h+2} b(i)$, where h is the length of the longest path in Q.*

Proof. By Lemma 2.9.2 we know that $m(I) \preceq Cm(I)$. Using Lemma 2.9.6 we have $C^{i-1} m(I) \preceq C^i m(I)$ and thus by transitivity $m(I) \preceq C^i m(I)$ for all $i \geq 1$. Thus it is enough to show that $m(I) \preceq C^{r_i} b(i)$ for some $0 \leq r_i \leq h + 2$.

We make case distinctions: First consider the case that $\alpha_{10} \geq 2$ or $0 \to t$. By Lemma 2.9.1 (ii) it follows that $b(0) \preceq Cb(0)$. Let t_i be the length of some path $0 \rightsquigarrow i$. We aim to show $b(0) \preceq C^{t_i} b(i)$. We know from the proof of Lemma 2.9.1 (i) that for $c \in \mathcal{K}$, $b(0) \preceq c$ if and only if $c_0 \geq 1$. Thus we need to show $(C^{t_i} b(i))_0 \geq 1$, but this follows directly from Lemma 2.9.5.

We further claim $m(I) \preceq Cb(0)$. Since we already know $b(0) \preceq Cb(0)$ by assumption, we only need to show $(Cb(0))_y \geq 1$ for $y \notin G$. By induction it is enough to show that for all $x \to y$ we have $(Cb(0))_y \geq (Cb(0))_x$. In order to prove this we observe

$$(Cb(0))_y = \alpha_{[0]y} + \sum_{j' \to y} \left((Cb(0))_{j'} - \delta_{[0]j'}\right) \alpha_{j'y} \geq \sum_{j' \to y} (Cb(0))_{j'} \geq (Cb(0))_x \,.$$

Summarizing, we have proven $m(I) \leq Cb(0) \leq CC^{t_i}b(i) = C^{t_i+1}b(i)$ and $t_i + 1 \leq h + 1 < h + 2$, so in this case we are done.

Now we consider the case $\alpha_{10} = 1$ and that there is no arrow $0 \to t$. We further consider the subcase that $\alpha_{x0} = 1$ for all $0 \to x$. Let $0 = x_0 \to x_1 \to \cdots \to x_s = t$ be a path different from the standard path $0 \to 1 \to \cdots \to t$. Since $\sum_{x \in I} \alpha_{x0} \geq 1$ by the grip properties, there must be a third neighbour of 0, say $0 \to y$. We make two claims:

(i) $b(x_1) + \cdots + b(x_s) \leq Cb(y)$

(ii) $m(I) \leq C\,(b(x_1) + \cdots + b(x_s))$

If these claims are correct, then we get $m(I) \leq C^2 b(y) \leq C^{t_i+2}b(i)$ by Lemma 2.9.7 with $t_i + 2 \leq h + 2$, which would finish the proof. We now prove (i): Set $c := Cb(y)$. We need to prove that $c_{x_i} \geq 1$ for all $i = 1, \ldots, s-1$ and that $c_t \geq c_{t-1} + 1$. By the second formula in Lemma 2.8.6 we have in general

$$c_j = -\delta_{yj} + \sum_{j \to j'} \delta_{yj'}\alpha_{j'j} + \sum_{j' \to j} c_{j'}\alpha_{j'j}.$$

Thus we get $c_{x_0} = c_0 = \alpha_{y0} = 1$. For $i \in \{1, \ldots, s\}$ we get

$$\begin{aligned} c_{x_i} &= -\delta_{yx_i} + \sum_{x_i \to j'} \delta_{yj'}\alpha_{j'x_i} + \sum_{j' \to x_i} c_{j'}\alpha_{j'x_i} \\ &\geq c_{x_{i-1}} - \delta_{yx_i} + \sum_{x_i \to j'} \delta_{yj'}\alpha_{j'x_i} + \sum_{j' \neq x_{i-1},\ j' \to x_i} c_{j'}\alpha_{j'x_i}. \end{aligned}$$

Now in case $y \neq x_i$ this is clearly greater or equal to c_{x_i-1}. If $y = x_i$, then $j' = 0$ provides one of the summands on the right, so the whole expression is also greater or equal to $c_{x_{i-1}}$. In any case we get $c_{x_i} \geq 1$ by induction. Note that in case $x_i = x_s = t$ we have $y \neq x_i$, and since $j' = t-1$ is one of the summands on the right side we get $c_t \geq 1 + c_{t-1}$, which proves (i).

For (ii) we need to show that for all $j \in I$ there is some $i \in \{1, \ldots, s\}$ such that $(Cb(x_i))_j \geq 1$. In general there is again the formula

$$(Cb(x_i))_j = -\delta_{x_ij} + \sum_{j \to j'} \delta_{x_ij'}\alpha_{j'j} + \sum_{j' \to j} (Cb(x_i))_{j'}\,\alpha_{j'j}.$$

In case $j \to x_i$ for some i we get $(Cb(x_i))_j \geq \alpha_{x_ij} \geq 1$. In case $j = t$ we get $(Cb(t))_t \geq -1 + (Cb(t))_{t-1} + (Cb(t))_{x_{s-1}}$, which is greater or equal to 1 by the case right before. In case that j does not map to any x_i and is unequal to t, we build the smallest path $j' \rightsquigarrow j$ such that j' maps to some x_i or is equal to t. If $j' \to x_i$ we compute $(Cb(x_i))_j = \sum_{j'' \to j} (Cb(x_i))_{j''}\,\alpha_{j''j} \geq (Cb(x_i))_{j''}$ for all $j'' \to j$, so we are done by induction since $(Cb(x_i))_{j'} \geq 1$ by the first case. In case $j' = t$ we compute $(Cb(t))_j = \sum_{j'' \to j} (Cb(t))_{j''}\,\alpha_{j''j} \geq (Cb(t))_{j''}$ for all $j'' \to j$, so we are also done by induction since $(Cb(t))_{j'} \geq 1$ by the second case. All in all this proves (ii) and therefore the subcase $\alpha_{x0} = 1$ for all $0 \to x$.

Now assume that there is an arrow $0 \to y$ such that $\alpha_{y0} \geq 2$. Since $\alpha_{10} = 1$ and $\alpha_{t0} = 0$ we have $y \notin G$. We claim $m(I) \leq Cb(y)$, which will finish the proof since $Cb(y) \leq C^{t_i+1}b(i)$ with $t_i+1 \leq h+2$ by Lemma 2.9.7. We need to show $(Cb(y))_j \geq 1$ for all

$j \in I$. For $j \to y$ we compute similarly as before $(Cb(y))_j = \alpha_{yj} + \sum_{j' \to j} (Cb(y))_{j'} \alpha_{j'j} \geq 1$. For $j = y$ we have $(Cb(y))_y = -1 + \sum_{j' \to y} (Cb(y))_{j'} \alpha_{j'y} \geq -1 + (Cb(y))_0 = -1 + \alpha_{y0} \geq 1$ and for the remaining $j \in I$ (those unequal to y that don't have an arrow to y) we get $(Cb(y))_j = \sum_{j' \to j} (Cb(y))_{j'} \alpha_{j'j}$, which is by induction greater or equal to 1. Therefore, we are done. □

Lemma 2.9.9. *For all $r = 0, \ldots, t+1$ we have $m(I) + \sum_{j=t-r+1}^{t} b(j) \leq C^r m(I)$.*

Proof. For $r = 0$ the statement is just $m(I) \leq m(I)$. We now do the induction step $r - 1 \to r$:

$$
\begin{aligned}
m(I) + \sum_{j=t-r+1}^{t} b(j) &= m(I) + \sum_{j=t-(r-1)+1}^{t} b(j) + b(t-r+1) \\
&\leq C^{r-1} m(I) + Cb(t-r+2) \\
&\ldots \\
&\leq C^{r-1} m(I) + C^{r-1} b(t) \\
&= C^{r-1} (m(I) + e(t)) \\
&\leq C^{r-1} C m(I) \\
&= C^r m(I),
\end{aligned}
$$

where we used repeatedly that $b(i-1) \leq Cb(i)$ according to Lemma 2.9.1 (i) and that $m(I) + e(t) \leq Cm(I)$ according to Lemma 2.9.2. □

Proposition 2.9.10. *The transformation C^{t+h+2} maps $\mathcal{K} \setminus \{0\}$ into $\mathring{\mathcal{K}}$, where again h is the length of the longest path in Q.*

Proof. Using Lemma 2.9.9 we see

$$
\sum_{i \in I} b(i) = b(0) + \sum_{x \notin G} b(x) + \sum_{j=1}^{t} b(j) = m(I) + \sum_{j=t-t+1}^{t} b(j) \leq C^t m(I).
$$

Together with Lemma 2.9.8 we get that all $j \in I$ satisfy $\sum_{i \in I} b(i) \leq C^{t+h+2} b(j) \in \mathcal{K}$. Since the $b(i)$ form a basis that means that $C^{t+h+2} b(j) = \sum_{i \in I} \lambda_i b(i)$ with $\lambda_i \geq 1$ for all $i \in I$. So $C^{t+h+2} b(j) \in \mathring{\mathcal{K}}$ for all $j \in I$, which proves the claim. □

2.10 Proof of Theorem 2.1.22

We now proof Theorem 2.1.22, still following [Rin94]. The idea is to apply some version of the Perron-Frobenius Theorem, see [Sen06, Theorem 1.1]:

Theorem 2.10.1 (Perron-Frobenius)**.** *Let $C \in \mathbb{R}^I$ be a non-negative square matrix (i.e. every entry is ≥ 0) such that some power C^s of C is strictly positive (i.e. every entry is > 0). Then*

(i) $\rho(C)$ is an eigenvalue of C with algebraic multiplicity one.

(ii) If $\rho(C) \neq \lambda \in \operatorname{spec}(C)$ then $|\lambda| < \rho(C)$.

Proof of Theorem 2.1.22. Let $A : \mathbb{R}^I \to \mathbb{R}^I$ be a connected, indefinite, symmetric generalized Cartan matrix as in the statement of the theorem. According to Proposition 2.7.13 we only need to consider the case that Q is a quiver for A such that (A, Q) has a grip.

Let $C := C(A, Q)$. Under these assumptions we constructed a basis $\{b_i\}_{i \in I}$ such that the cone $\mathcal{K} = \sum_{i \in I} \mathbb{R}_{\geq 0} b(i)$ gets mapped into itself under C by Corollary 2.8.11. We even showed that some power C^s of C sends $\mathcal{K} \setminus \{0\}$ into $\mathring{\mathcal{K}}$ in Proposition 2.9.10. When we represent C and C^s by matrices in the basis $\{b(i)\}_{i \in I}$ (which doesn't change eigenvalues), that means that C is a non-negative matrix and C^s is strictly positive. Thus by Theorem 2.10.1, $\rho(C)$ is an eigenvalue with multiplicity 1 and for all $\rho \neq \lambda \in \operatorname{spec}(C)$ we have $|\lambda| < \rho$. Furthermore, we already know from Proposition 2.2.8 that $\rho > 1$. That finishes the proof. □

2.11 Proof of Theorem 1.3.1

In this last section of chapter 2 we finish the proof of Theorem 1.3.1. Therefore, fix a connected wild quiver Q without oriented cycles and the associated path algebra $H = kQ$. Let $\Phi_H : \mathbb{C}^n \to \mathbb{C}^n$ be the associated Coxeter transformation, i.e. $\Phi_H = -C_H^t C_H^{-1}$ where C_H is the Cartan matrix of H (as mentioned, this is not a generalized Cartan matrix). This makes sense as soon as we have fixed an ordering $\{1, \ldots, n\}$ of the vertices of Q. Then by the reformulation in terms of generalized Cartan matrices in section 2.1 and since we proved Theorem 2.1.22 in the preceding section we already know that $\rho_H = \rho(\Phi_H) > 1$ is an eigenvalue of Φ_H of multiplicity one and that for all $\rho_H \neq \lambda \in \operatorname{spec}(\Phi_H)$ we have $|\lambda| < \rho_H$. It remains to prove the existence of strictly positive eigenvectors $x^+, x^- \in \mathbb{R}^n_{>0}$ such that $\Phi_H(x^+) = \rho_H x^+$ and $\Phi^{-1}(x^-) = \rho_H x^-$. First of all we want to make plausible why ρ_H is an eigenvalue of Φ_H^{-1} *at all*. To simplify notation, we omit the index H from now on, i.e. C is the Cartan matrix and Φ is the Coxeter transformation of H, ρ its spectral radius.

Proposition 2.11.1. Φ *and* Φ^{-1} *have the same characteristic polynomial* $\chi_\Phi = \chi_{\Phi^{-1}}$.

In particular, ρ is also the spectral radius of Φ^{-1}, itself an eigenvalue of Φ^{-1} and bigger than the absolute value of any other eigenvalue.

Proof. It is well known that $\chi_A = \chi_{A^t}$ for all square matrices A and that $\chi_{SAS^{-1}} = \chi_A$ whenever S is an invertible square matrix. We have

$$C\Phi^t C^{-1} = -CC^{-t}CC^{-1} = -CC^{-t} = \Phi^{-1}$$

and therefore:

$$\chi_\Phi = \chi_{\Phi^t} = \chi_{C\Phi^t C^{-1}} = \chi_{\Phi^{-1}},$$

proving the claim. □

The idea is now to use a third version of the Perron-Frobenius theorem to finish the proof of Theorem 1.3.1. We will construct some cone in which the eigenvectors x^+ and x^- will lie and use properties of Coxeter transformations developed in the last sections in order to show that they are strictly positive. In fact, we could have shown this already in the last sections, but we changed our matrices often by conjugation. Unfortunately, eigenvectors are *not* invariant under arbitrary conjugation, so we decided to be careful and follow the careful proofs in [dlPT90]. At one point we have to change the arguments since that paper only considers the bipartite case (i.e. every vertex is a sink or a source) and not arbitrary wild connected quivers.

Definition 2.11.2 (Solid cone). A *cone* in $\mathbb{R}^n$ is a subset $K \subseteq \mathbb{R}^n$ such that the following properties are satisfied:

(i) K is closed.

(ii) $K \cap (-K) = \{0\}$.

(iii) $K + K = K$, where $K + K$ is the set of sums $a + b$ with $a \in K$ and $b \in K$.

(iv) $\alpha K \subseteq K$ for all $\alpha \in \mathbb{R}_{\geq 0}$.

The cone K is called *solid* if additionally the following property is satisfied:

(v) The interior $\mathring{K}$ is nonempty.

The following theorem which we will use can be found in [Van68, Theorem 3.1]:

Theorem 2.11.3 (Perron-Frobenius for invariant Cones). *Let $K \subseteq \mathbb{R}^n$ be a solid cone and let $A \in M^{n\times n}(\mathbb{R})$ be a matrix which leaves K invariant, i.e. $\Phi(K) \subseteq K$. Then the following hold:*

(i) $\rho(A)$, the spectral radius of A, is an eigenvalue of A.

(ii) K contains an eigenvector corresponding to $\rho(A)$.

Now we define, as in [dlPT90], the preprojective cone:

Definition 2.11.4 (Preprojective cone). The *preprojective cone $K_{\mathcal{P}}$* of H is defined as

$$K_{\mathcal{P}} = \overline{\sum_{i=1}^{n} \sum_{r\in\mathbb{N}_{\geq 0}} \mathbb{R}_{\geq 0}\Phi^{-r}\underline{\dim}P(i)},$$

where $P(1), \dots, P(n)$ are as always the canonical indecomposable projective modules at the vertices of Q. In other words, since $\Phi^{-r}\underline{\dim}P(i) = \underline{\dim}\tau^{-t}P(i)$, the set $K_{\mathcal{P}}$ is just the topological closure of the sum of all the positive rays of dimension vectors of preprojective indecomposable modules.

Lemma 2.11.5. *$K_{\mathcal{P}}$ is a solid cone.*

Proof. By definition, $K_{\mathcal{P}}$ is closed, which shows (i). Define in this proof

$$k_{\mathcal{P}} = \sum_{i=1}^{n} \sum_{r\in\mathbb{N}_{\geq 0}} \mathbb{R}_{\geq 0}\Phi^{-r}\underline{\dim}P(i),$$

i.e. $K_{\mathcal{P}} = \overline{k_{\mathcal{P}}}$. Let $x \in k_{\mathcal{P}}$. Then x can be written as a non-negative linear combination of dimension vectors of (preprojective) modules. Since all these dimension vectors are non-negative we conclude $x \geq 0$. Therefore $x \in k_{\mathcal{P}} \cap (-k_{\mathcal{P}})$ if and only if $x = 0$. Now if $x \in K_{\mathcal{P}} \cap (-K_{\mathcal{P}})$, then x is the limit of a sequence of non-negative vectors (i.e. x is itself non-negative) and the limit of a sequence of non-positive vectors (i.e. x is itself non-positive) and therefore $x = 0$. This proves (ii). (iii) and (iv) are clear and since $k_{\mathcal{P}}$ has these properties and since addition and scalar multiplication are continuous operations. (v) Follows since the the cone $\sum_{i=1}^{n} \mathbb{R}_{\geq 0}\underline{\dim} P(i)$ is contained in $K_{\mathcal{P}}$ and since the vectors $\underline{\dim} P(i)$ form a basis of $\mathbb{R}^n$ (see chapter 1). □

Proposition 2.11.6. *There are vectors $x^+, x^- \in \mathbb{R}^n_{\geq 0}$ such that $\Phi(x^+) = \rho x^+$ and $\Phi^{-1}(x^-) = \rho x^-$.*

Proof. We consider only the statement about Φ^{-1}. For Φ, the same proof works, but considering the *preinjective cone*, which is clearly defined as

$$K_{\mathcal{I}} = \overline{\sum_{i=1}^{n} \sum_{r \in \mathbb{N}_{\geq 0}} \mathbb{R}_{\geq 0}\Phi^r \underline{\dim} I(i)}.$$

But back to Φ^{-1}: By continuity we have

$$\Phi^{-1} K_{\mathcal{P}} = \Phi^{-1}\overline{k_{\mathcal{P}}} \subseteq \overline{\Phi^{-1} k_{\mathcal{P}}},$$

where $k_{\mathcal{P}}$ is defined as in Lemma 2.11.5. Furthermore, we have

$$\Phi^{-1} k_{\mathcal{P}} = \Phi^{-1} \sum_{i=1}^{n} \sum_{r \in \mathbb{N}_{\geq 0}} \mathbb{R}_{\geq 0}\Phi^{-r} \underline{\dim} P(i) = \sum_{i=1}^{n} \sum_{r \in \mathbb{N}_{\geq 1}} \mathbb{R}_{\geq 0}\Phi^{-r} \underline{\dim} P(i) \subseteq k_{\mathcal{P}}$$

and therefore all in all $\Phi^{-1} K_{\mathcal{P}} \subseteq K_{\mathcal{P}}$. Since $K_{\mathcal{P}}$ is a solid cone, the Perron-Frobenius Theorem 2.11.3 guarantees that $K_{\mathcal{P}}$ contains an eigenvector x^- corresponding to the spectral radius of Φ^{-1}. But this spectral radius is by Proposition 2.11.1 just ρ, the spectral radius of Φ. Therefore, $\Phi^{-1} x^- = \rho x^-$. Since $x^- \in K_{\mathcal{P}}$, it follows from the proof of Lemma 2.11.5 that x^- is non-negative, so we are done. □

It remains to be shown that x^- and x^+ are in fact *strictly* positive and not just non-negative. We have $\Phi^{-1} x^- = \rho x^-$, which is equivalent to $\Phi x^- = \frac{1}{\rho} x^-$. We also have $\Phi x^+ = \rho x^+$. Furthermore we know $x^- \neq 0 \neq x^+$, since they are eigenvectors. In order to show that x^- and x^+ are strictly positive it is therefore enough to prove the following, slightly more general lemma:

Lemma 2.11.7. *Let $0 \neq y \geq 0$ and $\alpha > 0$ such that $\Phi y = \alpha y$. Then y is strictly positive, i.e. every entry is positive.*

Proof. We know that Φ is up to conjugation by a permutation matrix the same as $C(A, \mathrm{id})$ for a connected, indefinite, symmetric generalized Cartan matrix A, see Theorem 2.1.16. Conjugations by permutation matrices are the *good* conjugations in the sence that eigenvectors are only permuted, which means that positivity properties remain the same. Therefore we can assume $\Phi = C(A, \mathrm{id})$. Then by Proposition 2.6.3 we know that $\Phi = C(A, Q')$ for the quiver $Q' = Q(A, \mathrm{id})$ for A (The quiver Q' is the same as the quiver Q in $H = kQ$, except that all arrows are precisely reversed.). In particular, we can use the second formula in Lemma 2.8.6, since the proof did not use that (A, Q) has a grip. Assume that $y_j = 0$ for some $j \in \{1, \ldots, n\}$. Then we get (where the arrows are the ones in Q', not Q)

$$\begin{aligned} 0 = \alpha y_j = (\Phi y)_j &= -y_j + \sum_{j \to j'} y_{j'} \alpha_{j'j} + \sum_{j' \to j} (\Phi y)_{j'} \alpha_{j'j} \\ &= \sum_{j \to j'} y_{j'} \alpha_{j'j} + \sum_{j' \to j} y_{j'} (\alpha \alpha_{j'j}). \end{aligned}$$

All the appearing $\alpha_{j'j}$ are positive, as well as α. The $y_{j'}$ are non-negative. It follows $y_{j'} = 0$ for all neighbours j' of j. Since Q' is connected, we conclude by induction that $y = 0$, contradicting the assumption. Therefore, y must be strictly positive. This finishes the proof of the lemma and by the preceding discussion also the proof of Theorem 1.3.1. □

We end this chapter by looking at Example 2.1.24 again:

Example 2.11.8. Let the setup be as in Example 2.1.24. Then we already checked that Φ_H fulfils the conclusions of Theorem 2.1.22. We now check that also the eigenvector statements in Theorem 1.3.1 hold for this matrix. Remember that we have

$$\Phi = \Phi_H = \begin{pmatrix} -1 & 1 & 2 \\ -1 & 0 & 3 \\ -3 & 2 & 6 \end{pmatrix}$$

and that the eigenvalues are given by $-1, \rho$ and ρ^{-1}, where $\rho = 3 + 2\sqrt{2}$ is the spectral radius of Φ and $\rho^{-1} = 3 - 2\sqrt{2}$. Eigenvectors corresponding to the eigenvalues are given by

$$y = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}, \qquad x^- = \begin{pmatrix} 11 + 6\sqrt{2} \\ 6 + 2\sqrt{2} \\ 7 \end{pmatrix}, \qquad x^+ = \begin{pmatrix} 11 - 6\sqrt{2} \\ 6 - 2\sqrt{2} \\ 7 \end{pmatrix},$$

with $\Phi(y) = -y$, $\Phi(x^-) = \rho^{-1} x^-$ (i.e. $\Phi^{-1}(x^-) = \rho x^-$) and $\Phi(x^+) = \rho x^+$, which can be checked easily. We clearly see that both x^- and x^+ are strictly positive, which is exactly what is predicted by Theorem 1.3.1.

We note one further oddity of this example: The inverse of Φ is given by

$$\Phi^{-1} = \begin{pmatrix} 6 & 2 & -3 \\ 3 & 0 & -1 \\ 2 & 1 & -1 \end{pmatrix},$$

which is just the *point reflection* of Φ (i.e. for all $i, j \in \{1, 2, 3\}$ we have $\left(\Phi^{-1}\right)_{ij} = \Phi_{4-i,4-j}$). This is no coincidence: The generalized Cartan matrix A corresponding to the algebra $H = kQ$ is *point-symmetric*. Let $R_1, \ldots, R_n$ be the reflections corresponding to A (we write n instead of 3 to indicate that what we do works in general for point-symmetric A). Then we know that

$$\Phi = C(A, \mathrm{id}) = R_n \cdots R_1.$$

We have $R_i = R_i^{-1}$ for all i, see Lemma 2.1.7. Let $\pi : \{1, \ldots, n\} \to \{1, \ldots, n\}$ be the permutation $i \mapsto n + 1 - i$. Then by Proposition 2.2.4, Lemma 2.2.1, the fact that $P^\pi = (P^\pi)^{-1}$ and the fact that A is point-symmetric, the inverse of Φ is just given by

$$\Phi^{-1} = R_1 \cdots R_n = C(A, \pi) = P^\pi C(A^\pi, \mathrm{id})(P^\pi)^{-1} = C(A, \mathrm{id})^\pi = \Phi^\pi,$$

which means exactly what we just observed. We will later, after we will have developed the asymptotic behavior of dimension vectors, look at this example again. The symmetry we just developed will mean that for *symmetric* vectors x (i.e. vectors x such that $x_i = x_{n+1-i}$), the image $\Phi^{-1}(x)$ is just a just a mirrored version of $\Phi(x)$. This will produce a nice picture of a regular component in the Auslander-Reiten quiver of Q.

Chapter 3

Modules over wild path algebras

Let in this whole chapter always $H = kQ$ be a finite-dimensional quiver algebra, where Q is a wild connected quiver without oriented cycles. After our proof in chapter 2 we now have access to Theorem 1.3.1, which will serve as the main tool for developing the theory of finite-dimensional modules over H. In the first two sections, we develop the asymptotic behaviour of dimension vectors. In the third section we learn how to use defect functions associated to the eigenvectors x^+ and x^- in order to tell whether an indecomposable module is preprojective, regular or preinjective. This will lead to the completion of the proof of Theorem 1.3.2. In the two sections afterwards, we study morphisms between regular modules and will learn how regular components look like and illustrate this with an example. In the last section, we will see applications.

As always, all modules are finite-dimensional. Since we work in the whole chapter only with the fixed algebra H, we simplify the notation by omitting indices. For example we write $\operatorname{Hom}(X, Y)$ instead of $\operatorname{Hom}_H(X, Y)$. Since our algebra is hereditary, all higher Ext-groups disappear, so we will write $\operatorname{Ext}(X, Y)$ instead of $\operatorname{Ext}^1_H(X, Y)$. We also write $\dim$ instead of $\dim_k$ etc.

3.1 Asymptotic behaviour of preprojective and preinjective modules

In this section we develop – following [Ker96] – the asymptotic behaviour of dimension vectors Theorem 1.3.2 for the special case of preprojective and preinjective modules. First we have to develop some linear algebra that applies to the Coxeter transformation Φ corresponding to H and to its inverse Φ^{-1}.

Lemma 3.1.1. *Let $f \in \mathrm{GL}(\mathbb{C}^n)$ have an algebraically simple eigenvalue ρ such that $|\lambda| < \rho$ for all other eigenvalues $\lambda \in \operatorname{spec}(f)$. Let x be an eigenvector corresponding to ρ. Then there exists a subspace $W \subset \mathbb{C}^n$ of dimension $n-1$ such that $\mathbb{C}^n = \mathbb{C}x \oplus W$, $f(W) = W$ and for every $v = \alpha x + w \in \mathbb{C}^n$ (with $\alpha \in \mathbb{C}$ and $w \in W$) we get*

$$\lim_{k\to\infty} \frac{1}{\rho^k} f^k(v) = \alpha x.$$

Proof. Let χ_f be the characteristic polynomial of f. Since $\mathbb{C}$ is algebraically closed and since ρ is of multiplicity 1 we can write it as

$$\chi_f(X) = (X-\rho)\prod_{i=1}^{s}(X-\lambda_i)^{r_i},$$

where $\lambda_1,\ldots,\lambda_s$ are pairwise different eigenvalues of f with algebraic multiplicity r_i. Set $V_i := \ker(\lambda_i \operatorname{Id} - f)^{r_i}$, $V_\rho = \ker(\rho \operatorname{Id} - f)$ and $W := \bigoplus_{i=1}^{s} V_i$. Then by the Jordan normal form [Fis13, ch. 4.6] we have

(i) $f(V_\lambda) \subseteq V_\lambda$ for all $\lambda \in \{\rho, \lambda_1, \ldots, \lambda_s\}$,

(ii) $\mathbb{C}^n = V_\rho \oplus \bigoplus_{i=1}^{s} V_i$,

(iii) $f|_W : W \to W$ is of the form $f|_W = D + N$ where

  (a) D is diagonalizable with the same eigenvalues as $f|_W$,

  (b) N is nilpotent,

  (c) $D \circ N = N \circ D$.

Then, if x is an eigenvector corresponding to ρ we get $V_\rho = \mathbb{C}x$, since geometric multiplicity of eigenvalues is never bigger than algebraic multiplicity, see [Fis13, ch. 4.3]. Therefore we have $\mathbb{C}^n = \mathbb{C}x \oplus W$ by (ii). We have $f(W) \subseteq W$ by (i) and therefore $f(W) = W$ since f is an isomorphism.

Now let $v = \alpha x + w \in \mathbb{C}^n$ with $\alpha \in \mathbb{C}$ and $w \in W$. We get

$$\frac{1}{\rho^k} f^k(v) = \frac{\alpha}{\rho^k} f^k(x) + \frac{1}{\rho^k} f^k(w) = \alpha x + \frac{1}{\rho^k} f^k(w)$$

and this converges to αx if and only if $\lim_{k\to\infty} \frac{1}{\rho^k} f^k(w) = 0$. Since $\dim(W) = n-1$ and since N is nilpotent by (iii) (b) we have $N^{n-1} = 0$. Furthermore, since D has the same eigenvalues as $f|_W$ by (iii) (a), it is also invertible. Therefore we get for $k \geq n-2$, using (iii) (c) (which allows applying the binomial theorem),

$$\begin{aligned}
\frac{1}{\rho^k} f^k(w) &= \frac{1}{\rho^k}(D+N)^k(w) \\
&= \frac{1}{\rho^k}\sum_{i=0}^{n-2}\binom{k}{i} D^{k-i}N^i(w) \\
&= \sum_{i=0}^{n-2}\frac{1}{\rho^k}\binom{k}{i} D^k\left(D^{-i}N^i(w)\right).
\end{aligned}$$

Therefore it suffices to show that for all $i \in \{0,\ldots,n-2\}$ and all $w \in W$ (replacing $D^{-i}N^i(w) \in W$) we have

$$\lim_{k\to\infty} \frac{1}{\rho^k}\binom{k}{i} D^k(w) = 0.$$

Since D is diagonalizable we can write $w = w_1 + \cdots + w_{n-1}$ with $w_1,\ldots,w_{n-1}$ eigenvectors of D. Therefore it suffices to consider the case where w is itself an eigenvector of D, with eigenvalue $\lambda \in \{\lambda_1,\ldots,\lambda_s\}$. Then we have

$$\frac{1}{\rho^k}\binom{k}{i} D^k(w) = \frac{\lambda^k}{\rho^k}\binom{k}{i} w = \left(\frac{\lambda}{\rho}\right)^k \binom{k}{i} w.$$

Set $\mu := \frac{\lambda}{\rho}$. Then $|\mu| < 1$. Set $a_k := \mu^k \binom{k}{i}$. Then we get

$$\left|\frac{a_{k+1}}{a_k}\right| = |\mu| \cdot \frac{\binom{k+1}{i}}{\binom{k}{i}} = |\mu| \cdot \frac{k+1}{k+1-i} \to |\mu| < 1,$$

and therefore $\lim_{k\to\infty} \alpha_k = 0$, which is what we claimed. □

As always, denote by $P(1), \ldots, P(n)$ the indecomposable projective modules and by $I(1), \ldots, I(n)$ the indecomposable injective modules at the vertices $1, \ldots, n \in Q_0$.

Lemma 3.1.2. *(i) There is an irreducible morphism $P(i) \to P(j)$ if and only if there is an arrow $j \to i$ in Q. For all irreducible maps $X \to P(j)$ with X indecomposable we have $X = P(i)$ for some i.*

(ii) There is an irreducible morphism $I(i) \to I(j)$ if and only if there is an arrow $j \to i$ in Q. For all irreducible maps $I(i) \to Y$ with Y indecomposable we have $Y = I(j)$ for some j.

Proof. The sink map ending in $P(j)$ is of the form $\mathrm{rad}(P(j)) \to P(j)$ and we have

$$\mathrm{rad}(P(j)) = \bigoplus_{s(\alpha)=j} P(t(\alpha)).$$

Since irreducible morphisms ending in $P(j)$ are precisely the indecomposable direct summands of the sink map ending in $P(j)$, we proved (i).

The source map starting in $I(i)$ is of the form $I(i) \to I(i)/\operatorname{soc}(I(i))$ and we have

$$I(i)/\operatorname{soc}(I(i)) = \bigoplus_{t(\alpha)=i} I(s(\alpha)).$$

Since irreducible morphisms starting in $I(i)$ are precisely the indecomposable direct summands of the source map starting in $I(i)$, we proved (ii) □

Since $H = kQ$, where Q is wild, H is representation infinite, see [GR97, ch. 7]. Therefore, by [ARS97, ch. VIII Proposition 1.14], we get the following:

Lemma 3.1.3. *There is no indecomposable module in* $\mathrm{mod}(H)$ *which is both preprojective and preinjective.*

Lemma 3.1.4. *(i) The Auslander-Reiten sequence starting in $P = P(i)$ is of the form*

$$0 \longrightarrow P \longrightarrow \left(\bigoplus\nolimits_{t(\alpha)=i} P(s(\alpha))\right) \oplus \left(\bigoplus\nolimits_{s(\alpha)=i} \tau^- P(t(\alpha))\right) \longrightarrow \tau^- P \longrightarrow 0.$$

(ii) The Auslander-Reiten sequence ending in $I = I(i)$ is of the form

$$0 \longrightarrow \tau I \longrightarrow \left(\bigoplus\nolimits_{s(\alpha)=i} I(t(\alpha))\right) \oplus \left(\bigoplus\nolimits_{t(\alpha)=i} \tau I(s(\alpha))\right) \longrightarrow I \longrightarrow 0.$$

Proof. We only prove (ii), since (i) works dually. Let $X \to I$ be an irreducible map, where X is indecomposable. If X is injective, then $X = I(t(\alpha))$ for some $\alpha : i \to t(\alpha)$ by Lemma 3.1.2 (ii). If X is not injective, then $X = \tau Y$ and to $X \to I$ there corresponds an irreducible map $I \to Y$. Therefore, again by Lemma 3.1.2 we have $Y = I(s(\alpha))$ for an arrow $\alpha : s(\alpha) \to i$.

On the other hand, whenever we have an arrow $\alpha : i \to t(\alpha)$ we get a corresponding irreducible map $I(t(\alpha)) \to I$ again by Lemma 3.1.2. And if $\alpha : s(\alpha) \to i$ is an arrow, we get an irreducible map $I \to I(s(\alpha))$. Since $I(s(\alpha))$ is not projective by Lemma 3.1.3, there corresponds an irreducible map $\tau I(s(\alpha)) \to I$. All in all, the sink map looks as claimed. □

Remember that according to Theorem 1.3.1 there are strictly positive vectors x^+ and x^- in $\mathbb{R}^n$ such that $\Phi x^+ = \rho x^+$ and $\Phi^{-1} x^- = \rho x^-$, where ρ is the spectral radius of Φ and of Φ^{-1}.

Proposition 3.1.5. *Preprojective and preinjective modules show the following asymptotic behaviour:*

(i) If $P \neq 0$ is preprojective, then $\lim_{k\to\infty} \frac{1}{\rho^k} \underline{\dim} \tau^{-k} P = \alpha x^-$ with some $\alpha > 0$.

(ii) If $I \neq 0$ is preinjective, then $\lim_{k\to\infty} \frac{1}{\rho^k} \underline{\dim} \tau^{k} I = \beta x^+$ with some $\beta > 0$.

Proof. We prove only (ii), since (i) works dually by working with Φ^{-1} instead of Φ.

We first look at the special case that $I = I(i)$ is indecomposable injective at vertex $i \in Q_0$. By Theorem 1.3.1 and Lemma 3.1.1 there is a subspace $W \subseteq \mathbb{C}^n$ such that $\mathbb{C}^n = \mathbb{C}x^+ \oplus W$ and such that $\Phi(W) = W$. Therefore we can write $v := \underline{\dim} I = \beta x^+ + w$ with $\beta \in \mathbb{C}$ and $w \in W$. Then Lemma 3.1.1 as well as Lemma 1.1.3 tell us furthermore that

$$\lim_{k\to\infty} \frac{1}{\rho^k} \underline{\dim} \tau^k I = \lim_{k\to\infty} \frac{1}{\rho^k} \Phi^k(v) = \beta x^+.$$

Since we have $\frac{1}{\rho^k} \underline{\dim} \tau^k I \geq 0$ for all $k \in \mathbb{N}$ we also get $\beta x^+ \geq 0$ and therefore (since x^+ is strictly positive) we have $\beta \geq 0$.

Assume $\beta = 0$ which is equivalent to saying that $\lim_{k\to\infty} \frac{1}{\rho^k} \underline{\dim} \tau^k I = 0$. Take the Auslander-Reiten sequence

$$0 \longrightarrow \tau I \longrightarrow \left(\bigoplus I(j)\right) \oplus \left(\bigoplus \tau I(j')\right) \longrightarrow I \longrightarrow 0,$$

from Lemma 3.1.4. We conclude, using additivity of dimension vectors on short exact sequences:

$$\begin{aligned}
0 &= \beta x^+ \\
&= \lim_{k\to\infty} \frac{1}{\rho^k} \Phi^{k-1} \underline{\dim} \tau I \\
&= \lim_{k\to\infty} \frac{1}{\rho^k} \Phi^{k-1} \left[\underline{\dim} \left[\left(\bigoplus I(j) \right) \oplus \left(\bigoplus \tau I(j') \right) \right] - \underline{\dim} I \right] \\
&= \lim_{k\to\infty} \frac{1}{\rho^k} \Phi^{k-1} \left[\underline{\dim} \left[\left(\bigoplus I(j) \right) \oplus \left(\bigoplus \tau I(j') \right) \right] \right] - \frac{1}{\rho} \lim_{k\to\infty} \frac{1}{\rho^{k-1}} \Phi^{k-1}(v)
\end{aligned}$$

$$= \lim_{k\to\infty} \left[\sum_j \frac{1}{\rho^k} \Phi^{k-1} \underline{\dim} I(j) + \sum_{j'} \frac{1}{\rho^k} \Phi^{k-1} \underline{\dim} \tau I(j') \right] - \frac{1}{\rho} \beta x^+$$
$$= \sum_j \frac{1}{\rho} \lim_{k\to\infty} \frac{1}{\rho^{k-1}} \underline{\dim} \tau^{k-1} I(j) + \sum_{j'} \lim_{k\to\infty} \frac{1}{\rho^k} \underline{\dim} \tau^k I(j').$$

All the summands $\frac{1}{\rho^k}\underline{\dim}\tau^{k-1}I(j)$ and $\frac{1}{\rho^k}\underline{\dim}\tau^k I(j')$ are non-negative. Therefore, since the sum is zero and since j and j' run through all neighbours of i by Lemma 3.1.4 we get that for all neighbours l of i we have $\lim_{k\to\infty}\frac{1}{\rho^k}\underline{\dim}\tau^k I(l) = 0$. We summarize: From $\lim_{k\to\infty}\frac{1}{\rho^k}\underline{\dim}\tau^k I(i) = 0$ we concluded that $\lim_{k\to\infty}\frac{1}{\rho^k}\underline{\dim}\tau^k I(l) = 0$ for all neighbours l of i. Using induction and that Q is connected we conclude that $\lim_{k\to\infty}\frac{1}{\rho^k}\underline{\dim}\tau^k I(j) = 0$ for all $j \in Q_0$. By construction this means $\underline{\dim} I(j) \in W$ for all $j \in Q_0$. But this is not possible since the $\underline{\dim} I(j)$ form a basis of $\mathbb{Z}^n$ (and therefore also $\mathbb{C}^n$) and since $W \subsetneq \mathbb{C}^n$. Therefore, the assumption that $\beta = 0$ leads to a contradiction and since $\beta \geq 0$ we indeed have $\beta > 0$.

Now let I more generally be indecomposable preinjective. Then $I = \tau^s I(i)$ for some $i \in Q_0$. Therefore we get

$$\lim_{k\to\infty} \frac{1}{\rho^k} \underline{\dim} \tau^k I = \rho^s \lim_{k\to\infty} \frac{1}{\rho^{k+s}} \underline{\dim} \tau^{k+s} I(i) = \rho^s \beta x^+,$$

with $\beta > 0$ and therefore also $\rho^s\beta > 0$. Now if $I = \bigoplus_j I_j$ is a general nonzero preinjective module, where I_j are indecomposable preinjective, we get

$$\lim_{k\to\infty} \frac{1}{\rho^k} \underline{\dim} \tau^k I = \sum_j \lim_{k\to\infty} \frac{1}{\rho^k} \underline{\dim} \tau^k I_j = \sum_j \beta_j x^+,$$

where the individual β_j satisfy $\beta_j > 0$. Therefore we have $\beta := \sum_j \beta_j > 0$. □

Thus we have shown Theorem 1.3.2 partly, but only considering preprojective and preinjective modules. We need to study regular modules more thoroughly to get the full Theorem.

3.2 Asymptotic behaviour of regular modules

In order to prove the asymptotic behaviour of dimension vectors in the regular case, we first need some results on the homomorphisms between regular modules over our wild algebra $H = kQ$. We still follow [Ker96].

Lemma 3.2.1. *For any vector $x \in \mathbb{Z}^n$ there is an integer N such that the following holds: For all $s \geq N$, all regular modules Y with $\underline{\dim}(Y) \leq x$, all regular modules R and all $f \in \mathrm{Hom}(\tau^s Y, R)$, the module $\ker(f)$ is regular.*

Also the dual statement is true. For any vector $x \in \mathbb{Z}^n$ there is an integer N such that the following holds: For all $s \geq N$, all regular modules Y with $\underline{\dim} Y \leq x$, all regular modules R and all $f \in \mathrm{Hom}(R, \tau^{-s} Y)$, the module $\mathrm{coker}(f)$ is regular.

Proof. By Proposition 3.1.5, there is an N such that $\underline{\dim}\tau^{-s}P(i) > x$ for all $s \geq N$ and all $i = 1, \dots, n$. Take s, Y, R and $f : \tau^s Y \to R$ as in the formulation of this lemma. Let $K = \ker(f)$ and $I = \operatorname{im}(f)$. We get a short exact sequence

$$0 \to K \to \tau^s Y \to I \to 0.$$

By Corollary B.1.3 we know that K does not have a nontrivial preinjective summand and I does not have a nontrivial preprojective summand. As I furthermore injects in the regular R, I also does not contain nontrivial preinjective summands, and so I is regular. As all three modules in the short exact sequence thus do not contain any preinjective direct summands, Proposition 1.1.1 assures that the sequence will stay short exact after applying τ^{-s}:

$$0 \to \tau^{-s}K \to Y \to \tau^{-s}I \to 0$$

Therefore and by definition of N we get $\underline{\dim}\tau^{-s}K \leq \underline{\dim}Y \leq x < \underline{\dim}\tau^{-m}P(i)$ for all indecomposable projective modules $P(i)$ and all $m \geq N$.

We already know that K does not contain preinjective direct summands. In order to conclude that it is regular, we have to show that it also does not contain preprojective direct summands. Assume otherwise, i.e. $K = \tau^{-t}P(i) \oplus K'$ for some t, i and K'. Then by additivity of τ^- (so $\tau^{-s}K = \tau^{-(s+t)}P(i) \oplus \tau^{-s}K'$) we get

$$\underline{\dim}\tau^{-(s+t)}P(i) \leq \underline{\dim}\tau^{-s}K < \underline{\dim}\tau^{-m}P(i).$$

This holds for all $m \geq N$, therefore especially for $m = s + t$, which is absurd. So we conclude that K is regular.

The proof of the dual statement works just dually. □

For what comes next we need the notion of *elementary modules*, the building blocks of regular modules. For the definition and basic facts, see the appendix B.2.

Proposition 3.2.2. *Let E be indecomposable regular. Then the following are equivalent:*

(i) E is elementary.

(ii) There exists an integer N such that for all $s \geq N$, $\tau^s E$ has no nontrivial regular factor modules.

(iii) If $Y \neq 0$ is a regular submodule of E, then E/Y is preinjective.

Those three statements are also all equivalent to the duals of (ii) and (iii):

(ii') There exists an integer N such that for all $s \geq N$, $\tau^{-s}E$ has no nontrivial regular submodule.

(iii') If $Y \neq 0$ is a regular quotient module of E, i.e. there is a surjection $f : E \to Y$, then $\ker(f)$ is preprojective.

Proof. If $\tau^l E$ has no nontrivial regular factor module, then it is elementary and so by Proposition B.2.2 we get that $E = \tau^{-l}\tau^l E$ is elementary as well. Thus (ii) implies (i).

Since nontrivial regular modules are not preinjective, (iii) also implies (i).

We now prove that (i) implies (ii): Since E is regular we get by Lemma 3.2.1 that there is an integer N such that for all $s \geq N$, all regular modules R and all $f \in \mathrm{Hom}(\tau^s E, R)$ the module $\ker(f)$ is regular itself. Now assume $\tau^s E$ has a nontrivial regular factor module R, so we get a surjection $f : \tau^s E \to R$. Then $\ker(f) \neq 0$ is regular, so the short exact sequence $0 \to \ker(f) \to \tau^s E \to R \to 0$ contradicts the fact that $\tau^s E$ is elementary by Proposition B.2.2.

Finally we prove that (i) implies (iii): Assume to the contrary of (iii) that there is $0 \neq Y$ regular such that $E/Y = Z_1 \oplus Z_2$ with $Z_1 \neq 0$ regular and Z_2 preinjective (There cannot be a nontrivial preprojective direct summand because of Proposition B.1.2). We get the following commutative exact diagram:

$$\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
 & & 0 & & & & Z_2 & & \\
 & & \downarrow & & & & \downarrow & & \\
0 & \longrightarrow & Y & \longrightarrow & E & \longrightarrow & E/Y & \longrightarrow & 0 \\
 & & \downarrow & & \| & & \downarrow & & \\
0 & \longrightarrow & K & \longrightarrow & E & \longrightarrow & Z_1 & \longrightarrow & 0 \\
 & & \downarrow & & & & \downarrow & & \\
 & & Z_2 & & & & 0 & & \\
 & & \downarrow & & & & & & \\
 & & 0 & & & & & &
\end{array}$$

In the diagram, the map $E \to Z_1$ is the composition $E \to E/Y = Z_1 \oplus Z_2 \to Z_1$ and K is its kernel. Since Y is clearly also in the kernel, we get the inclusion $Y \to K$ which has by Snake Lemma cokernel Z_2. K is a submodule of the regular module E and thus has by Corollary B.1.3 no nontrivial preinjective direct summand. From Lemma B.1.4 (i) we know that K does also not have a preprojective direct summand (since Y and Z_2 do not have such summands), thus K is regular. Thus the lower short exact sequence shows that E is not elementary and we are done.

That (i), (ii') and (iii') are all equivalent can be proven dually. □

Lemma 3.2.3. *There is no indecomposable module* $X \neq 0$ *allowing an integer* $m \neq 0$ *with* $\underline{\dim}\tau^m X = \underline{\dim} X$.

Proof. Assume $\underline{\dim}\tau^m X = \underline{\dim} X$ and $m \geq 1$. Then X can clearly not be preprojective and therefore by Proposition 1.1.3 (i) we have $\underline{\dim}\tau^i X = \Phi^i \underline{\dim} X$ for all $i \geq 0$. Define $x := \sum_{i=0}^{m-1} \Phi^i \underline{\dim} X$. Then we compute

$$\Phi(x) = \sum_{i=0}^{m-1} \Phi^{i+1} \underline{\dim} X = \Phi^m \underline{\dim} X + \sum_{i=1}^{m-1} \Phi^i \underline{\dim} X = \sum_{i=0}^{m-1} \Phi^i \underline{\dim} X = x.$$

Let q be the Tits form corresponding to our algebra $H = kQ$, as defined in the introduction. Then we get using Proposition 1.2.2 that for all $y \in \mathbb{Z}^n$

$$\langle x, y \rangle = -\langle y, \Phi(x) \rangle = -\langle y, x \rangle$$

and therefore

$$(x, y) := \langle x, y\rangle + \langle y, x\rangle = 0.$$

By [Sch14, Proposition 8.5] we conclude that q is positive-semidefinite (i.e. Q is Euclidean), contrary to q being indefinite. Therefore the assumption cannot be true. That proves the case $m \geq 1$. The case $\underline{\dim}\tau^m X = \underline{\dim} X$ for $m \leq -1$ is proven dually by considering $x := \sum_{i=m+1}^{0} \Phi^i \underline{\dim} X$. □

The idea for the proof of the following lemma is based on the proof of [Bae86, Lemma 1.1]. We also need some tilting theory which we develop in Appendix B.3.

Lemma 3.2.4. *Let X be a nonzero regular module. Then there exists $t \neq 0$ such that* $\mathrm{Hom}(X, \tau^t X) \neq 0$.

Proof. We only need to show this for X indecomposable. By Lemma 3.2.3 we know that the modules $\{X, \tau^2 X, \dots, \tau^{2n} X\}$ are pairwise nonisomorphic, where n is the number of vertices in the quiver Q. Furthermore, because they are $n + 1$ vectors, the set $\{\underline{\dim}\tau^{2i} X\}_{i\in\{0,\dots,n\}}$ is linearly dependent in $\mathbb{Z}^n$. Thus by Proposition B.3.3 we get

$$\mathrm{Ext}\left(\bigoplus_{i=0}^{n} \tau^{2i} X, \bigoplus_{i=0}^{n} \tau^{2i} X\right) \neq 0$$

and thus there are $i, j \in \{0, \dots, n\}$ such that $\mathrm{Ext}\left(\tau^{2i} X, \tau^{2j} X\right) \neq 0$. By the Auslander-Reiten formula Theorem 1.1.2 this means that $\mathrm{Hom}(\tau^{2j} X, \tau^{2i+1} X) \neq 0$ and therefore by Proposition 1.1.1 we get $\mathrm{Hom}(X, \tau^{2(i-j)+1} X) \neq 0$, where clearly $t = 2(i - j) + 1 \neq 0$. □

Lemma 3.2.5. *Let E be an elementary module. Then the following statements (dual to each other) hold:*

(i) There are $s, t \in \mathbb{Z}$, $t \neq 0$ and a preinjective module $0 \neq Q_E$ such that there exists a short exact sequence

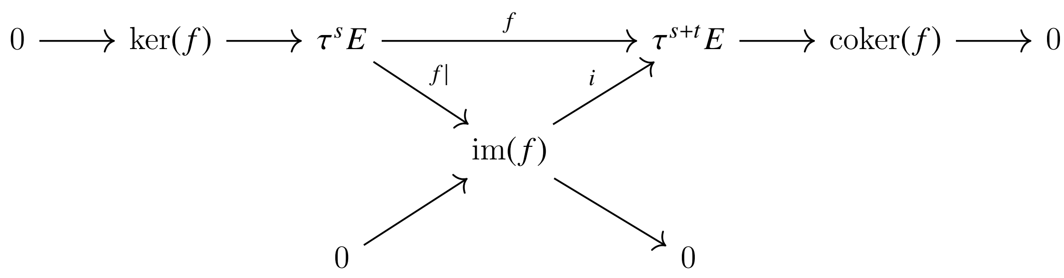
$$0 \longrightarrow \tau^s E \longrightarrow \tau^{s+t} E \longrightarrow Q_E \longrightarrow 0.$$

(ii) There are $s, t \in \mathbb{Z}$, $t \neq 0$ and a preprojective module $0 \neq P_E$ such that there is a short exact sequence

$$0 \longrightarrow P_E \longrightarrow \tau^{-(s+t)} E \longrightarrow \tau^{-s} E \longrightarrow 0$$

Proof. We only prove (i) since (ii) works dually by using all the dual ingredients. By Lemma 3.2.4 we find $0 \neq t \in \mathbb{Z}$ such that $\mathrm{Hom}(E, \tau^t E) \neq 0$.

By Lemma 3.2.2 there is an $s \in \mathbb{Z}$ such that $\tau^s E$ has no nontrivial regular factor module. We have $0 \neq \mathrm{Hom}(E, \tau^t E) \cong \mathrm{Hom}(\tau^s E, \tau^{s+t} E)$ and can therefore find $0 \neq f : \tau^s E \to \tau^{s+t} E$. We get the following diagram:



By Corollary B.1.3 (*iii*) we get that $\mathrm{im}(f)$ is regular and furthermore $\tau^{s+t}E$ is elementary by Proposition B.2.2 (*i*). Therefore by Lemma 3.2.2 (*iii*) we get that $Q_E := \mathrm{coker}(f)$ is preinjective. f is injective since otherwise $\mathrm{im}(f)$ is a nontrivial regular factor module of $\tau^s E$, but that cannot be by the choice of s. Furthermore, $\mathrm{im}(f) \neq \tau^{s+t}E$, since otherwise f would be surjective, thus an isomorphism, which contradicts that $\tau^s E \not\cong \tau^{s+t}E$ by Lemma 3.2.3. Therefore $Q_E \neq 0$. All in all we are done. □

Proposition 3.2.6. *Let $X \neq 0$ regular. Then we get:*

(i) There is $\lambda_X^- > 0$ such that $\lim_{k\to\infty} \frac{1}{\rho^k}\underline{\dim}\tau^{-k}X = \lambda_X^- x^-$.

(ii) There is $\lambda_X^+ > 0$ such that $\lim_{k\to\infty} \frac{1}{\rho^k}\underline{\dim}\tau^{k}X = \lambda_X^+ x^+$.

Proof. We only prove (*ii*), since (*i*) works dually, again using all the dual ingredients. First consider the case where X is elementary. Then by Lemma 3.2.5 (*i*) there are $s, t \in \mathbb{Z}$ and a preinjective module $0 \neq Q_X$ together with a short exact sequence $0 \to \tau^s X \to \tau^{s+t}X \to Q_X \to 0$. Applying τ^k for $k \geq 0$ we get a short exact sequence $0 \to \tau^{k+s}X \to \tau^{k+s+t}X \to \tau^k Q_X \to 0$. Now by Proposition 3.1.5 (*ii*) there is $\beta > 0$ such that $\lim_{k\to\infty} \frac{1}{\rho^k}\underline{\dim}\tau^k Q_X = \beta x^+$. By Lemma 3.1.1 we can write $\underline{\dim}\tau^{s+t}X = \beta_{(\tau^{s+t}X)}x^+ + w$ with $\beta_{(\tau^{s+t})X} \in \mathbb{C}$ and $w \in W$ and get

$$\begin{aligned}
\beta x^+ &= \lim_{k\to\infty} \frac{1}{\rho^k}\underline{\dim}\tau^k Q_X \\
&\leq \lim_{k\to\infty} \frac{1}{\rho^k}\underline{\dim}\tau^{k+s+t}X \\
&= \lim_{k\to\infty} \frac{1}{\rho^k}\Phi^k\underline{\dim}\tau^{s+t}X \\
&= \beta_{(\tau^{s+t}X)}x^+.
\end{aligned}$$

$\beta_{(\tau^{s+t}X)}x^+ \geq \beta x^+$ shows - since x^+ is strictly positive - that $\beta_{(\tau^{s+t}X)} \geq \beta > 0$. We get

$$\begin{aligned}
\lim_{k\to\infty} \frac{1}{\rho^k}\underline{\dim}\tau^k X &= \lim_{k\to\infty} \frac{1}{\rho^{k+s+t}}\underline{\dim}\tau^{k+s+t}X \\
&= \frac{1}{\rho^{s+t}} \lim_{k\to\infty} \frac{1}{\rho^k}\underline{\dim}\tau^{k+s+t}X \\
&= \left(\frac{1}{\rho^{s+t}}\beta_{(\tau^{s+t}X)}\right) x^+.
\end{aligned}$$

Since $\lambda_X^+ := \frac{1}{\rho^{s+t}}\beta_{(\tau^{s+t}X)} > 0$, we are done in the case X elementary.

Now let X be a general regular module. Then by Proposition B.2.2 there is a filtration $X = X_0 \supset X_1 \supset \cdots \supset X_r \supset X_{r+1} = 0$ such that all factor modules $E_i := X_i/X_{i+1}$ for $i \in \{0, \ldots, r\}$ are elementary. We conclude

$$\begin{aligned}
\lim_{k\to\infty} \frac{1}{\rho^k}\underline{\dim}\tau^k X = \lim_{k\to\infty} \frac{1}{\rho^k}\Phi^k\underline{\dim}X &= \sum_{i=0}^{r} \lim_{k\to\infty} \frac{1}{\rho^k}\Phi^k\underline{\dim}E_i \\
&= \sum_{i=0}^{r} \lim_{k\to\infty} \frac{1}{\rho^k}\underline{\dim}\tau^k E_i = \sum_{i=0}^{r} \lambda_{E_i}^+ x^+,
\end{aligned}$$

where we have $\sum_{i=0}^{r} \lambda_{E_i}^+ > 0$, since the summands are > 0. This finishes the proof. □

3.3 Defect functions and classification of modules

In the last two sections, we proved the asymptotic behaviour of dimension vectors. In order to also prove statement (*iii*) of Theorem 1.3.2, we need to classify preprojective, regular and preinjective modules via the defect functions $\langle x^-, - \rangle : \mathbb{R}^n \to \mathbb{R}$ and $\langle -, x^+ \rangle : \mathbb{R}^n \to \mathbb{R}$, where x^- and x^+ are strictly positive eigenvectors of the Coxeter transformation corresponding to ρ^{-1} and ρ, respectively. We follow [dlPT90] and use the same conventions as before.

Lemma 3.3.1. *Let $x \in \mathbb{Z}^n$. Then we have*

$$\left\langle \underline{\dim} P(i), x \right\rangle = \left\langle x, \underline{\dim} I(i) \right\rangle = x_i$$

for all $i \in \{1, \ldots, n\}$.

Proof. Let C be the Cartan matrix of the algebra H. It has as columns the dimension vectors of the indecomposable projective modules. Therefore

$$\left\langle \underline{\dim} P(i), x \right\rangle = \left(\underline{\dim} P(i)\right)^t C^{-t} x = \left(C^{-1} \underline{\dim} P(i)\right)^t x = e(i)^t x = x_i.$$

The Cartan matrix has as rows also the dimension vectors of the indecomposable injective modules, see Chapter 1. Therefore we get

$$\left\langle x, \underline{\dim} I(i) \right\rangle = x^t C^{-t} \underline{\dim} I(i) = x^t e(i) = x_i.$$

This finishes the proof. □

Theorem 3.3.2. *Let X be an indecomposable module. Then we have the following:*

(i) X is preprojective if and only if $\left\langle x^-, \underline{\dim} X \right\rangle < 0$. Moreover, if X is not preprojective, then $\left\langle x^-, \underline{\dim} X \right\rangle > 0$.

(ii) X is preinjective if and only if $\left\langle \underline{\dim} X, x^+ \right\rangle < 0$. Moreover, if X is not preinjective then $\left\langle \underline{\dim} X, x^+ \right\rangle > 0$.

(iii) X is regular if and only if both $\left\langle x^-, \underline{\dim} X \right\rangle > 0$ and $\left\langle \underline{\dim} X, x^+ \right\rangle > 0$.

Proof. Statement (*iii*) follows directly from (*i*) and (*ii*). Moreover, (*ii*) follows from the dual of the proof of (*i*), since x^+ relates to the preinjective cone in the same way as x^- relates to the preprojective cone as defined in section 2.11. Therefore we only prove (*i*):

Let X be a preprojective module. Then $\tau^m X = P(i)$ for some i. We get using Proposition 1.2.2 and Lemma 3.3.1, the fact that x^- is strictly positive and using $\Phi^{-1} x^- = \rho x^-$:

$$\begin{aligned} \left\langle x^-, \underline{\dim} X \right\rangle &= \left\langle x^-, \Phi^{-m-1} \Phi \underline{\dim} P(i) \right\rangle \\ &= \left\langle \Phi^{m+1} x^-, -\underline{\dim} I(i) \right\rangle \\ &= -\rho^{-m-1} \left\langle x^-, \underline{\dim} I(i) \right\rangle \\ &= -\rho^{-m-1} x_i^- < 0 \end{aligned}$$

Now let on the other hand X be an indecomposable module such that $\left\langle x^-, \underline{\dim} X \right\rangle < 0$. x^- lies by the proof of Proposition 2.11.6 in the preprojective cone $K_{\mathcal{P}} = \overline{k_{\mathcal{P}}}$. Therefore, there is a sequence of vectors $(u_m)_{m\in\mathbb{N}}$ where $u_m \in k_{\mathcal{P}}$ and such that $\lim_{m\to\infty} u_m = x^-$. By definition of $k_{\mathcal{P}}$, we have

$$u_m = \sum_{i=1}^{l_m} \mu_i^{(m)} \underline{\dim} V_i^{(m)},$$

where $V_i^{(m)}$ are preprojective modules and $\mu_i^{(m)} > 0$ (we can assume that the sum is non-trivial since $x^- \neq 0$). Since the homological bilinear form is given by a continuous matrix multiplication, we have

$$0 > \left\langle x^-, \underline{\dim} X \right\rangle = \lim_{m\to\infty} \left\langle u_m, \underline{\dim} X \right\rangle,$$

and therefore there is some $m \in \mathbb{N}$ such that

$$0 > \left\langle u_m, \underline{\dim} X \right\rangle = \sum_{i=1}^{l_m} \mu_i^{(m)} \left\langle \underline{\dim} V_i^{(m)}, \underline{\dim} X \right\rangle.$$

Since all $\mu_i^{(m)}$ are positive, there is therefore some $i \in \{1, \ldots, l_m\}$ such that

$$0 > \left\langle \underline{\dim} V_i^{(m)}, \underline{\dim} X \right\rangle = \dim \operatorname{Hom}\left(V_i^{(m)}, X\right) - \dim \operatorname{Ext}\left(V_i^{(m)}, X\right).$$

Therefore we get using the Auslander-Reiten formula Theorem 1.1.2:

$$0 \neq \operatorname{Ext}\left(V_i^{(m)}, X\right) \cong D \operatorname{Hom}\left(X, \tau V_i^{(m)}\right)$$

Using Proposition B.1.2 we see this is only possible if X is preprojective itself.

Now let X be an indecomposable module which is not preprojective. First consider the case that X is preinjective. We therefore have $X = \tau^m I(i)$ for an indecomposable injective module $I(i)$. Then using the same tools as in the case where X was preprojective we get

$$\begin{aligned} \left\langle x^-, \underline{\dim} X \right\rangle &= \left\langle x^-, \Phi^m \underline{\dim} I(i) \right\rangle \\ &= \left\langle \Phi^{-m} x^-, \underline{\dim} I(i) \right\rangle \\ &= \rho^m \left\langle x^-, \underline{\dim} I(i) \right\rangle \\ &= \rho^m x_i^- > 0. \end{aligned}$$

Finally consider the case that X is regular. Since X is not preprojective, we already know that $\left\langle x^-, \underline{\dim} X \right\rangle \geq 0$. Assume that $\left\langle x^-, \underline{\dim} X \right\rangle = 0$. Assume further that X has minimal dimension among all regular modules with this property. We claim that X does not have any nontrivial regular submodule:

Assume $0 \neq Y \subseteq X$ is regular. Let $C = X/Y$ and consider the short exact sequence

$$0 \longrightarrow Y \longrightarrow X \longrightarrow C \longrightarrow 0.$$

By Proposition B.1.2 we know that all indecomposable summands of C must be regular or preinjective. Therefore, and by what we already showed, we get

$$\begin{aligned} 0 \leq \left\langle x^-, \underline{\dim} Y \right\rangle &= \left\langle x^-, \underline{\dim} X \right\rangle - \left\langle x^-, \underline{\dim} C \right\rangle \\ &= -\left\langle x^-, \underline{\dim} C \right\rangle \leq 0. \end{aligned}$$

Therefore, $\langle x^-, \underline{\dim} Y\rangle = 0$, so by the minimality of X among all regular modules with this propery we get $Y = X$. This shows that X does not have any nontrivial regular submodule.

Now we know from Lemma 3.2.4 that there is $0 \neq f \in \operatorname{Hom}(\tau^{-m}X, X)$ for some $0 \neq m \in \mathbb{Z}$. $\operatorname{im}(f)$ is a quotient of $\tau^{-m}X$ and a submodule of X and thus by Proposition B.1.2 it must be regular itself. Therefore, since X does not have nontrivial submodules, we get $\operatorname{im}(f) = X$ and therefore f is surjective. On the other hand, f cannot be an isomorphism by Lemma 3.2.3, therefore it is a proper epimorphism. If we define $K = \ker(f) \neq 0$ then we get the exact sequence

$$0 \longrightarrow K \longrightarrow \tau^{-m}X \xrightarrow{f} X \longrightarrow 0.$$

Write $K = K_p \oplus K_r$ with K_p preprojective and K_r regular. We know that the composition

$$K_r \longrightarrow K_p \oplus K_r = K \longrightarrow \tau^{-m}X$$

is a proper monomorphism. By exactness of τ on regular modules, also the map

$$\tau^m K_r \longrightarrow X$$

is a proper monomorphism. But we showed above that X does not contain nontrivial regular submodules, and therefore $\tau^m K_r = 0$. It follows $K_r = 0$ and we conclude $0 \neq K = K_p$. Hence, by what we have already shown, we get

$$\begin{aligned} 0 \geq \langle x^-, \underline{\dim} K\rangle &= \langle x^-, \underline{\dim}\tau^{-m}X\rangle - \langle x^-, \underline{\dim} X\rangle \\ &= \langle x^-, \Phi^{-m}\underline{\dim} X\rangle \\ &= \langle \Phi^m x^-, \underline{\dim} X\rangle \\ &= \rho^{-m}\langle x^-, \underline{\dim} X\rangle = 0, \end{aligned}$$

a contradiction. We conclude that the assumption that $\langle x^-, \underline{\dim} X\rangle = 0$ led to a contradiction and therefore we indeed have $\langle x^-, \underline{\dim} X\rangle > 0$. This finishes the proof. □

We end this section by finally finishing the proof of Theorem 1.3.2 and afterwards continuing our example:

Proof of Theorem 1.3.2. Let X be, as in (i), a nonzero module without indecomposable preinjective direct summands. Then we can write $X = X_p \oplus X_r$ with X_p preprojective and X_r regular. Using Proposition 3.1.5 (i) and 3.2.6 (i) we get $\alpha > 0$ and $\lambda^-_{X_r} > 0$ such that

$$\lim_{t\to\infty} \frac{1}{\rho^t}\underline{\dim}\tau^{-t}X = \lim_{t\to\infty} \frac{1}{\rho^t}\underline{\dim}\tau^{-t}X_p + \lim_{t\to\infty} \frac{1}{\rho^t}\underline{\dim}\tau^{-t}X_r = \alpha x^- + \lambda^-_{X_r} x^- = (\alpha + \lambda^-_{X_r})x^-.$$

Since $\lambda^-_X := \alpha + \lambda^-_{X_r} > 0$, we proved (i). For (ii) we proceed in the same way, using Proposition 3.1.5 (ii) and 3.2.6 (ii).

For (iii) we proceed as follows: Let X without indecomposable preinjective direct summands and Y without indecomposable preprojective direct summands. Then we get

$$\begin{aligned}\lim_{t\to\infty}\frac{1}{\rho^t}\left\langle \underline{\dim}\tau^{-t}X, \underline{\dim}Y\right\rangle &= \left\langle \lim_{t\to\infty}\frac{1}{\rho^t}\underline{\dim}\tau^{-t}X, \underline{\dim}Y\right\rangle\\ &= \left\langle \lambda_X^- x^-, \underline{\dim}Y\right\rangle\\ &= \lambda_X^-\left\langle x^-, \underline{\dim}Y\right\rangle > 0,\end{aligned}$$

which can be seen by decomposing Y into indecomposable non-preprojective summands and using Theorem 3.3.2 (i). We further get, using that $\tau : \mathrm{mod}(H)_p \to \mathrm{mod}(H)_i$ is an equivalence with inverse τ^-:

$$\begin{aligned}\left\langle \underline{\dim}\tau^{-t}X, \underline{\dim}Y\right\rangle &= \dim \mathrm{Hom}(\tau^{-t}X, Y) - \dim \mathrm{Ext}(\tau^{-t}X, Y)\\ &= \dim \mathrm{Hom}(X, \tau^t Y) - \dim \mathrm{Ext}(X, \tau^t Y)\\ &= \left\langle \underline{\dim}X, \underline{\dim}\tau^t Y\right\rangle,\end{aligned}$$

which shows the equality of the two limits in (iii). This finishes the proof. □

Example 3.3.3. We continue Example 2.1.24 and 2.11.8 by demonstrating Theorem 3.3.2. Let Q as in the examples before, $H = kQ$, Φ the Coxeter transformation and

$$x^- = \begin{pmatrix} 11+6\sqrt{2}\\ 6+2\sqrt{2}\\ 7\end{pmatrix}, \qquad x^+ = \begin{pmatrix} 11-6\sqrt{2}\\ 6-2\sqrt{2}\\ 7\end{pmatrix}$$

the eigenvectors for ρ^{-1} and ρ, respectively, where ρ is the spectral radius of Φ. The dimension vectors of the three canonical simple modules of Q are just the standard basis vectors, so it will be easy to test via our classification theorem if they are preinjective, preprojective or regular. Note that $S(1) = P(1)$ and $S(3) = I(3)$, so we already know a priori that they are projective and injective, respectively. We expect $S(2)$ to be regular for *symmetry reasons* (the vertex 2 sits exactly in the middle of the quiver and this does not change by turning all arrows arround, which can be realized by applying the duality $D : \mathrm{mod}(H) \to \mathrm{mod}(H^{\mathrm{op}}) = \mathrm{mod}(k(Q^{\mathrm{op}})) \cong \mathrm{mod}(H)$). Now we check all of this. Remember that $\langle -,-\rangle$ is given explicitly by $\langle x, y\rangle = x^t C_H^{-t} y$ (see the introduction), where

$$C_H^{-t} = \begin{pmatrix} 1 & 1 & 3\\ 0 & 1 & 1\\ 0 & 0 & 1\end{pmatrix}^{-t} = \begin{pmatrix} 1 & 0 & 0\\ -1 & 1 & 0\\ -2 & -1 & 1\end{pmatrix}.$$

Therefore we can just compute:

$$\left\langle x^-, \underline{\dim}S(1)\right\rangle = (x^-)^t C_H^{-t}\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix} 11+6\sqrt{2} & 6+2\sqrt{2} & 7\end{pmatrix}\begin{pmatrix}1\\-1\\-2\end{pmatrix} = -9+4\sqrt{2} < 0,$$

so $S(1)$ is preprojective (in fact, as said before, even projective). We further have

$$\left\langle \underline{\dim}S(3), x^+\right\rangle = \begin{pmatrix}0 & 0 & 1\end{pmatrix} C_H^{-t} x^+ = \begin{pmatrix}-2 & -1 & 1\end{pmatrix}\begin{pmatrix} 11-6\sqrt{2}\\ 6-2\sqrt{2}\\ 7\end{pmatrix} = -21+14\sqrt{2} < 0,$$

so $S(3)$ is preinjective (in fact even injective, as said before). For $S(2)$ we compute

$$\langle x^-, \underline{\dim} S(2)\rangle = (x^-)^t C_H^{-t}\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}11+6\sqrt{2} & 6+2\sqrt{2} & 7\end{pmatrix}\begin{pmatrix}0\\1\\-1\end{pmatrix} = -1+2\sqrt{2} > 0,$$

so $S(2)$ is not preprojective. Finally we have

$$\langle \underline{\dim} S(2), x^+\rangle = \begin{pmatrix}0 & 1 & 0\end{pmatrix} C_H^{-t} x^+ = \begin{pmatrix}-1 & 1 & 0\end{pmatrix}\begin{pmatrix}11-6\sqrt{2}\\6-2\sqrt{2}\\7\end{pmatrix} = -5+4\sqrt{2} > 0,$$

so $S(2)$ is also not preinjective and therefore regular. We end this part of the example by demonstrating the asymptotic behaviour of $S(2)$ in τ-direction. The asymptotic behaviour in the other direction follows then just by symmetry since – as mentioned in the last part of the example – Φ^{-1} is just a permuted version of Φ. Let $S = \begin{pmatrix}y & x^- & x^+\end{pmatrix}$ be the change of coordinates where y is the eigenvector of the third eigenvalue -1 (when considered as a row vector: $y = \begin{pmatrix}1 & -2 & 1\end{pmatrix}$), as computed before. Let Φ' be the diagonalized version of Φ, i.e.

$$\Phi' = \operatorname{diag}(-1, \rho^{-1}, \rho) = \begin{pmatrix}-1 & 0 & 0\\0 & \rho^{-1} & 0\\0 & 0 & \rho\end{pmatrix}.$$

Then we have $\Phi = S\Phi' S^{-1}$ and therefore

$$\begin{aligned}\lim_{t\to\infty}\frac{1}{\rho^t}\underline{\dim}\,\tau^t S(2) &= \lim_{t\to\infty}\frac{1}{\rho^t}\Phi^t\underline{\dim} S(2)\\ &= \lim_{t\to\infty}\frac{1}{\rho^t}S(\Phi')^t S^{-1}e(2)\\ &= S\lim_{t\to\infty}\frac{1}{\rho^t}\operatorname{diag}((-1)^t, \rho^{-t}, \rho^t)S^{-1}e(2)\\ &= S\lim_{t\to\infty}\operatorname{diag}((-1/\rho)^t, \rho^{-2t}, 1)S^{-1}e(2).\end{aligned}$$

One can compute that

$$S^{-1} = \begin{pmatrix}\frac{1}{8} & \frac{-3}{8} & \frac{1}{8}\\ \frac{-1+5\sqrt{2}}{112} & \frac{3-\sqrt{2}}{112} & \frac{1-\sqrt{2}}{16}\\ \frac{-1-5\sqrt{2}}{112} & \frac{3+\sqrt{2}}{112} & \frac{1+\sqrt{2}}{16}\end{pmatrix}.$$

Therefore and since $\rho > 1$ (i.e. $\lim_{t\to\infty}(-1/p)^t = \lim_{t\to\infty}\rho^{-2t} = 0$) we get

$$\begin{aligned}\lim_{t\to\infty}\frac{1}{\rho^t}\underline{\dim}\,\tau^t S(2) &= S\lim_{t\to\infty}\operatorname{diag}\left((-1/\rho)^t, \rho^{-2t}, 1\right)\begin{pmatrix}\frac{-3}{8}\\ \frac{3-\sqrt{2}}{112}\\ \frac{3+\sqrt{2}}{112}\end{pmatrix}\\ &= S\left(\frac{3+\sqrt{2}}{112}e(3)\right)\\ &= \frac{3+\sqrt{2}}{112}x^+,\end{aligned}$$

which demonstrates Proposition 3.2.6 with $\lambda_X^+ = \frac{3+\sqrt{2}}{112} > 0$. We will in section 3.5, after proving how regular components look like, continue this example.

3.4 Morphisms between regular modules

In order to see in the next section how the regular components of the Auslander-Reiten quiver of the wild algebra $H = kQ$ look like, we first have to prove some facts about morphisms between regular modules. We still follow [Ker96].

Lemma 3.4.1. *For a regular module X the τ-orbit $\{\tau^m X \mid m \in \mathbb{Z}\}$ is infinite and for any natural number N we get $\dim_k \tau^m X > N$ for $|m| \gg 0$.*

Proof. This follows directly from Lemma 3.2.3, since for $|m|$ big, we can achieve arbitrarily large entries in the dimension vector of $\underline{\dim}\tau^m X$, which leads to arbitrarily large k-dimension. □

Lemma 3.4.2. *Let X, Y be modules with $\mathrm{Hom}(X, Y) \neq 0$. If X and Y have filtrations $X = X_0 \supset \cdots \supset X_{r+1} = 0$, $Y = Y_0 \supset \cdots \supset Y_{s+1} = 0$, then there are i and j with $\mathrm{Hom}(X_i/X_{i+1}, Y_j/Y_{j+1}) \neq 0$.*

Proof. Let $0 \neq f : X \to Y$ and i the unique index with $f|_{X_i} \neq 0$ but $f|_{X_{i+1}} = 0$. Then we get a nonzero map $g := \overline{f|_{X_i}} : X_i/X_{i+1} \to Y$. Now let j be the unique index such that $\mathrm{im}(g) \subseteq Y_j$ but $\mathrm{im}(g) \nsubseteq Y_{j+1}$. Then the composition $X_i/X_{i+1} \to Y_j \to Y_j/Y_{j+1}$ of g and the canonical projection is nonzero and thus yields the result. □

Lemma 3.4.3. *Let X and Y be regular modules. Then there is an integer N such that $\mathrm{Hom}(\tau^m X, Z) = 0$ for all $m \geq N$ and all regular modules Z with $\dim_k Z \leq \dim_k Y$.*

Proof. We can assume that X and Y are nonzero. We first additionally assume that X is elementary. By Proposition 3.2.2 there exists an N_1 such that for all $m \geq N_1$, $\tau^m X$ has no nontrivial regular factor module. Then if $f : \tau^m X \to R$ is a nonzero homomorphism with R regular, f has to be injective: Indeed, as $\mathrm{im}(f)$ is both a factor module of $\tau^m X$ and a submodule of R, it is regular by Corollary B.1.3. So $f : \tau^m X \to \mathrm{im}(f)$ must be an isomorphism, since $\tau^m X$ does not have any nontrivial factor module.

By Lemma 3.4.1 there is an $N_2 \in \mathbb{Z}$ such that $\dim_k \tau^m X > \dim_k Y$ for all $m \geq N_2$. For $m \geq N = \max(N_1, N_2)$ we get the result: Let Z be a regular module with $\dim_k Z \leq \dim_k Y$. Then $\dim_k Z < \dim_k \tau^m X$. So there cannot be an injective homomorphism $f : \tau^m X \to Z$ and so by what we have shown above, every f: $\tau^m X \to Z$ is zero.

Now let X be a general regular module. Then we have a filtration $X = X_0 \supset \cdots \supset X_{r+1} = 0$ with $E_i = X_i/X_{i+1}$ elementary. For each i there is an integer N_i such that $\mathrm{Hom}(\tau^m E_i, Z) = 0$ for all $m \geq N_i$ and all regular modules Z with $\dim_k Z \leq \dim_k Y$. Let $N = \max\{N_i\}$. Then by Lemma 3.4.2 (using the filtrations $\tau^m X = \tau^m X_0 \supset \cdots \supset \tau^m X_{r+1} = 0$ and $Z \supset 0$) we see that for all $m \geq N$, $\mathrm{Hom}(\tau^m X, Z) = 0$. □

Corollary 3.4.4. *(i) If $X \neq 0$ is regular and $m > 0$, there are neither injective nor surjective maps in $\mathrm{Hom}(X, \tau^{-m} X)$.*

(ii) Let X be any module, Y nonzero regular and $f : X \to Y$ be injective. Then $\mathrm{Hom}(\tau Y, X)$ does not contain a monomorphism.

(iii) Let X be nonzero regular, Y any module and $f : X \to Y$ be surjective. Then $\mathrm{Hom}(Y, \tau^- X)$ does not contain an epimorphism.

Proof. we first prove (i): Assume that there is a monomorphism (epimorphism) $f : X \to \tau^{-m}X$. Since τ is an equivalence on regular modules, all the maps $\tau^{sm}f : \tau^{sm}X \to \tau^{(s-1)m}X$ are also monomorphisms (epimorphisms) for $s \geq 0$, and so is the composition

$$\tau^{sm}X \to \tau^{(s-1)m}X \to \cdots \to X,$$

which is in particular nonzero. If we set $Y = Z = X$ in Lemma 3.4.3, we get a contradiction.

For (ii), assume there is a monomorphism $g : \tau Y \to X$. Then the composition $f \circ g : \tau Y \to X \to Y$ is a monomorphism. Applying τ^{-}, we get a monomorphism $Y \to \tau^{-1}Y$, contradicting (i).

For (iii), assume there is an epimorphism $g : Y \to \tau^{-}X$. Then the composition $g \circ f : X \to Y \to \tau^{-1}X$ is an epimorphism, again contradicting (i). □

3.5 Structure of the regular components

As before, we follow [Ker96]. In this section we will see that the regular components in the Auslander-Reiten quiver are of the form $\mathbb{Z}A_\infty$, which will be explained soon. It looks like an infinite net bordered by certain *quasi-simple* modules.

Lemma 3.5.1. *Let $g : E \to X$ be an irreducible map between regular modules. Then all translates $\tau^m g : \tau^m E \to \tau^m X$ are also irreducible.*

Proof. Assume that $\tau^m g$ factors:

$$\begin{array}{ccc} \tau^m E & \xrightarrow{\quad \tau^m g \quad} & \tau^m X \\ & {\scriptstyle (g_i^1)} \searrow \quad \nearrow {\scriptstyle (g_i^2)^T} & \\ & Z_1 \oplus Z_2 \oplus Z_3 & \end{array}$$

Here we assume Z_1 to be preprojective, Z_2 to be regular and Z_3 to be preinjective. Then, since $\tau^m E$ and $\tau^m X$ are regular we get that $g_1^1 = 0 = g_3^2$ by Proposition B.1.2. Therefore $\tau^m g$ factors as $\tau^m g = g_2^2 \circ g_2^1$ through the regular module Z_2. By applying τ^{-m} to the factorization we get a corresponding factorization of g:

$$\begin{array}{ccc} E & \xrightarrow{\quad g \quad} & X \\ & {\scriptstyle \tau^{-m}(g_2^1)} \searrow \quad \nearrow {\scriptstyle \tau^{-m}(g_2^2)} & \\ & \tau^{-m} Z_2 & \end{array}$$

The map g is irreducible, so we deduce that $\tau^{-m}\left(g_2^1\right)$ is a split monomorphism or $\tau^{-m}\left(g_2^2\right)$ is a split epimorphism. Since τ^{-m} is an equivalence on regular modules, g_2^1 is a split monomorphism or g_2^2 is a plit epimorphism. Thus $\left(g_i^1\right)$ is a split monomorphism or $\left(g_i^2\right)^T$ is a split epimorphism, and so $\tau^m g$ is irreducible. □

Lemma 3.5.2. *Let $0 \to \tau X \xrightarrow{f} E \xrightarrow{g} X \to 0$ be the Auslander-Reiten sequence of the regular module X. Then for all $m \in \mathbb{Z}$, the sequence $0 \to \tau^{m+1}X \xrightarrow{\tau^m f} \tau^m E \xrightarrow{\tau^m g} \tau^m X \to 0$ is an Auslander-Reiten sequence of $\tau^m X$.*

Proof. Since τ^m is an equivalence on regular modules, the new sequence is still short exact. The result follows since by Lemma 3.5.1, both $\tau^m f$ and $\tau^m g$ are irreducible. □

Theorem 3.5.3. *Let X be an indecomposable regular module and let*

$$0 \longrightarrow \tau X \xrightarrow{(f_i)} \bigoplus_{i=1}^{r} E_i \xrightarrow{(g_i)^T} X \longrightarrow 0$$

be an Auslander-Reiten sequence with E_i indecomposable. Then we get the following:

(i) If $r \geq 2$, then $g = (g_1, g_2) : E_1 \oplus E_2 \to X$ is surjective.

(ii) At most one of the f_i is injective.

(iii) At most one of the f_i is surjective.

(iv) In fact, $r \leq 2$ always holds. In case $r = 2$ we can arrange the indices in such a way that f_1 and g_2 are surjective and f_2 and g_1 are injective, which shows $\underline{\dim} E_1 < \underline{\dim} X$ and $\underline{\dim} \tau X < \underline{\dim} E_2$.

Proof. We prove (i): g is the direct summand of a sink map and thus irreducible. Then g is either injective or surjective. We assume it is injective. Then by left exactness and additivity of τ also the map $\tau g = (\tau g_1, \tau g_2) : \tau E_1 \oplus \tau E_2 \to \tau X$ is injective.

We look at the Auslander-Reiten sequences $0 \to \tau E_i \to \tau X \oplus Z_i \to E_i \to 0$ for suitable Z_i (τX has to occur in the middle term, since the irreducible map $f_i : \tau X \to E_i$ has to occur in the sink map). We get $\underline{\dim} \tau X < \underline{\dim} (E_i \oplus \tau E_i)$. Using this and that g and τg are both injective, we get

$$2\underline{\dim} \tau X \leq \underline{\dim} (E_1 \oplus E_2) + \underline{\dim} (\tau E_1 \oplus \tau E_2) < \underline{\dim} X + \underline{\dim} \tau X,$$

hence $\underline{\dim} \tau X < \underline{\dim} X$. In fact, since by Lemma 3.5.2 all the translated sequences are also Auslander-Reiten sequences, we get by induction that all the maps $\tau^i g$ are injective, and so repetition of the same argument yields $\underline{\dim} X > \underline{\dim} \tau X > \underline{\dim} \tau^2 X > \ldots$, which cannot be true.

For (ii), assume f_1 and f_2 are both injective. By Corollary 3.4.4 there is thus no monomorphism in $\mathrm{Hom}(\tau E_i, \tau X)$ for $i = 1, 2$. Since τ is left exact, also $\mathrm{Hom}(E_i, X)$ cannot contain monomorphisms, and so the irreducible maps $g_i : E_i \to X$ have to be surjective for $i = 1, 2$. We get

$$\underline{\dim} (\tau X \oplus X) < \underline{\dim} (E_1 \oplus E_2) \leq \underline{\dim} \tau X + \underline{\dim} X,$$

were we use in the first step that f_1 is injective and g_2 is surjective and in the second step the additivity of dimension vectors on short exact sequences. We get a contradiction.

For (iii), assume that both f_1 and f_2 are surjective. By the right exactness of τ on regular modules, all maps $\tau^i f_1$ and $\tau^i f_2$ are surjective as well. By (i) we know there is an epimorphism $E_1 \oplus E_2 \to X$, and by composition with $f_1 \oplus f_2$ we get an epi $\tau(X^2) = (\tau X)^2 \to X$. By applying τ and using right exactness, we thus get an epi $\tau^2(X^2) \to \tau X$ and so (by building the direct sum of maps and composing with the earlier epimorphism) a chain $\tau^2\left(X^{2^2}\right) = \tau^2(X^4) \to \tau(X^2) \to X$. Repeating this argument, we get an epimorphism $\tau^N\left(X^{2^N}\right) \to X$ for all $N \geq 0$, which is nonzero. But by setting $Y = Z = X$, this is a contradiction to Lemma 3.4.3.

Finally, we prove (iv): Since all f_i are irreducible and thus injective or surjective, and since by (ii) and (iii) at most one of them is injective respectively surjective, we conclude that $r \leq 2$.

Now assume $r = 2$. We may assume that f_1 is surjective and f_2 is injective. We claim that g_1 is injective: Since f_1 is surjective, by Corollary 3.4.4 (iii) there is no epimorphism $E_1 \to \tau^-\tau X = X$, so g_1 must be a monomorphism, since it is irreducible. We also claim that g_2 is surjective: Since f_2 is injective, there is no monomorphism $\tau E_2 \to \tau X$ by Corollary 3.4.4 (ii), and so by left exactness of τ there is also no monomorphism $E_2 \to X$, proving that g_2 has to be surjective.

Injectivity of g_1 (and the fact that g_1 cannot be an isomorphism, since Auslander-Reiten sequences do not split) gives $\underline{\dim} E_1 < \underline{\dim} X$ and injectivity of f_2 gives in the same way $\underline{\dim} \tau X < \underline{\dim} E_2$. □

Definition 3.5.4 (Quasi-simple module). Let X be an indecomposable regular module. It is called *quasi-simple* if the middle term of the Auslander-Reiten sequence $0 \to \tau X \to E \to X \to 0$ is indecomposable.

Lemma 3.5.5. *Let C be a regular component in the Auslander-Reiten quiver and X indecomposable with minimal k-dimension in C, then X and all translates $\tau^m X$ for $m \in \mathbb{Z}$ are quasi-simple.*

Notice that such X obviously exists.

Proof. If X was not quasi-simple, we would be in case $r = 2$ in the terminology of Theorem 3.5.3. But then by (iv) we get $\underline{\dim} X > \underline{\dim} E_1$, thus $\dim_k X > \dim_k E_1$, contradicting the minimality of the k-dimension of X.

Then for $\tau^m X$ we can construct an Auslander-Reiten sequence by applying τ^m to an Auslander-Reiten sequence of X by Lemma 3.5.2. In particular, the middle term is also indecomposable and so $\tau^m X$ is also quasi-simple. □

Lemma 3.5.6. *Let C be a regular component in the Auslander-Reiten quiver and let X be quasi-simple in it. Then the following hold:*

(i) There is an infinite chain of irreducible monomorphisms

$$X = X(1) \to X(2) \to X(3) \to \cdots \to X(n) \to \ldots,$$

and it is unique up to isomorphism of diagrams.

(ii) There is an infinite chain of irreducible epimorphisms

$$\cdots \to [n]X \to \cdots \to [3]X \to [2]X \to [1]X = X$$

and it is unique up to isomorphism of diagrams.

Proof. We prove (i), (ii) is done dually. X is quasi-simple and so by Lemma 3.5.2, $\tau^- X$ is quasi-simple as well. We thus get an Auslander-Reiten sequence

$$0 \to X = X(1) \to X(2) \to \tau^- X \to 0$$

with $X(2)$ indecomposable and $X(1) \to X(2)$ an irreducible monomorphism. Since $X(2) \to \tau^- X$ is irreducible, it is part of the first map of an Auslander-Reiten sequence starting in $X(2)$. But since it is not injective, the middle term must consist of exactly two indecomposable summands by Theorem 3.5.3:

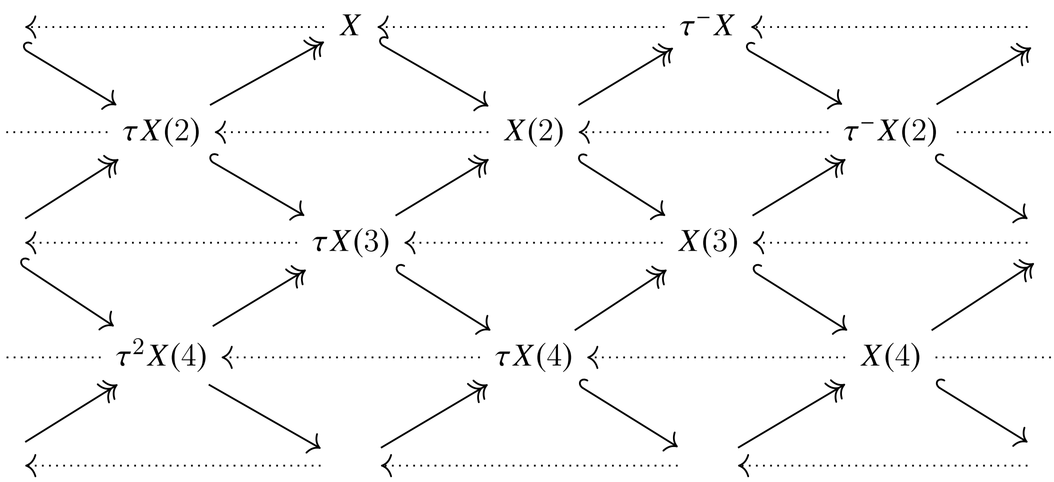
$$0 \to X(2) \to \tau^- X \oplus X(3) \to \tau^- X(2) \to 0,$$

where $X(2) \to X(3)$ is an irreducible monomorphism. The map $X(3) \to \tau^- X(2)$ is not injective by theorem 3.5.3 and the injectivity of $\tau^- X \to \tau^- X(2)$. Inductively, we get the desired sequence. It is unique since the irreducible monomorphisms must be part of an Auslander-Reiten sequence, and these are unique up to isomorphism. □

Lemma 3.5.7. *It holds* $\tau^m(X(i)) = (\tau^m X)(i)$ *and* $[i](\tau^m X) = \tau^m([i]X)$ *for all quasi-simple modules* X *and all* $m \in \mathbb{Z}$. *We therefore can just write* $\tau^m X(i)$ *for these modules.*

Proof. The chain $X = X(1) \to X(2) \to X(3) \to \ldots$ only consists of irreducible monomorphisms of regular modules. Since τ is an equivalence on regular modules and preserves irreducible maps by Lemma 3.5.1, the chain $(\tau^m X)(1) = \tau^m(X(1)) \to \tau^m(X(2)) \to \tau^m(X(3)) \ldots$ consists only of irreducible monomorphisms as well. By the uniqueness part of Lemma 3.5.6 the result follows. A similar proof works for the chain of irreducible epimorphisms. □

Theorem 3.5.8. *Let* C *be a regular component in the Auslander-Reiten quiver of* A *and let* X *be a quasi-simple module in it. Then* C *is of type* $\mathbb{Z}A_\infty$. *More precisely, it looks as follows, spreading infinitely to the left, right and downwards:*



In particular, for $i \geq 2$ *there is an Auslander-Reiten sequence*

$$0 \to \tau^m X(i) \to \tau^{m-1} X(i-1) \oplus \tau^m X(i+1) \to \tau^{m-1} X(i) \to 0.$$

Furthermore, the sequence $\ldots [3]X \to [2]X \to [1]X = X$ *from Lemma 3.5.7 is up to isomorphism the same as the sequence* $\ldots \tau^2 X(3) \to \tau X(2) \to X(1) = X$.

Proof. The arrows in down-right direction are the infinite sequences of irreducible monomorphisms from Lemma 3.5.6. We get the dotted arrows in left-direction (indicating the τ-translation) by Lemma 3.5.7. From the theory of Auslander-Reiten sequences we know that between $\tau^m X(i)$ and $\tau^{m-1}X(i)$ there has to be an Auslander-Reiten sequence. Let $i \geq 2$. Since we already have an irreducible map $\tau^{m-1}X(i-1) \to \tau^{m-1}X(i)$, the module $\tau^{m-1}X(i-1)$ has to occur in the middle term of this Auslander-Reiten sequence, which gives an irreducible map $\tau^m X(i) \to \tau^{m-1}X(i-1)$. From Theorem 3.5.3 we know that the middle term of an Auslander Reiten sequence cannot have more then 2 indecomposable summands and that the arrows in up-right direction have to be surjective. From the proof of Lemma 3.5.6 we know that $0 \to \tau^m X \to \tau^m X(2) \to \tau^{m-1}X \to 0$ is an Auslander-Reiten sequence, so in the component C there cannot be any new arrows besides the one we already found. This proves that C is of the desired form, except that we still need to show that the picture does not overlap itself:

Assume $\tau^k X(i) \cong \tau^{k'}X(i')$ for some $k, k' \in \mathbb{Z}$ and some $i, i' \geq 1$. Then $\tau^{k-k'}X(i) \cong X(i')$. Therefore we have without loss of generality $k' = 0$. We want to show $k = 0$ and $i = i'$. If $i' = 1$ then $X(i')$ is quasi-simple and therefore $\tau^k X(i)$ is quasi-simple as well, which is by construction only possible if $i = 1$. Then we get $X(1) \cong \tau^k X(1)$ and therefore $k = 0$ by Lemma 3.2.3. If $i' > 1$, both $X(i')$ and $\tau^k X(i)$ are not quasi-simple and therefore there are irreducible monomorphisms

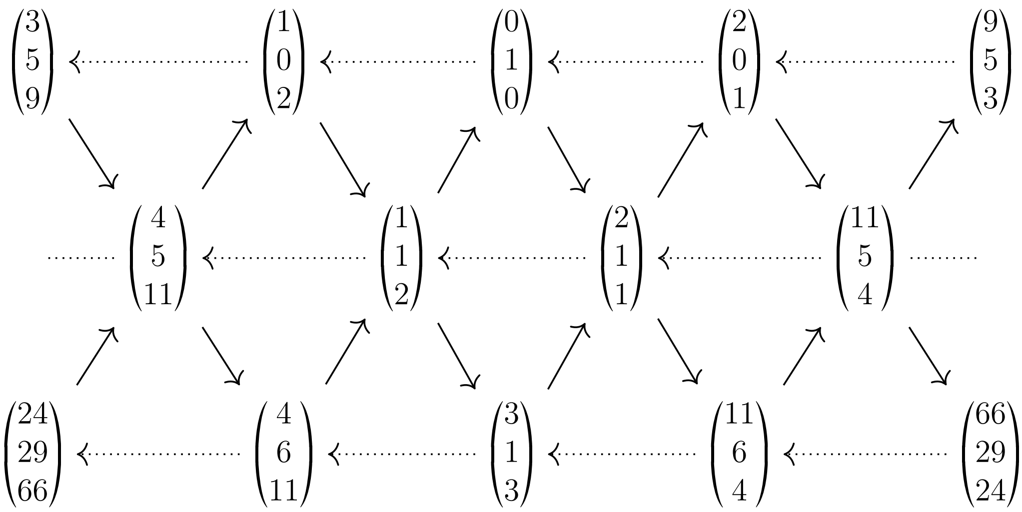
$$\begin{array}{ccc} X(i'-1) & \hookrightarrow & X(i') \\ & & \downarrow \cong \\ \tau^k X(i-1) & \hookrightarrow & \tau^k X(i). \end{array}$$

By the uniqueness of irreducible monomorphisms we conclude $X(i'-1) \cong \tau^k X(i-1)$ and therefore $i' = i$ and $k = 0$ by induction. The other statements are clear. □

Example 3.5.9. We continue here our earlier example 2.1.24, 2.11.8 and 3.3.3. Remember that the simple module $S(2)$ is regular and that the Coxeter transformation Φ and its inverse (just a mirror version, as we saw before) are given by

$$\Phi = \begin{pmatrix} -1 & 1 & 2 \\ -1 & 0 & 3 \\ -3 & 2 & 6 \end{pmatrix}, \qquad \Phi^{-1} = \begin{pmatrix} 6 & 2 & -3 \\ 3 & 0 & -1 \\ 2 & 1 & -1 \end{pmatrix}.$$

Since $S(2)$ is necessarily of minimal k-dimension in its regular component (it is of dimension one) it is quasi-simple by Lemma 3.5.5. Therefore, Theorem 3.5.8 tells us how the regular component looks relative to $S(2)$. We indicate this using dimension vectors (note that dimension vectors do in general not determine modules). We can compute the dimension vectors of the τ-translates of $S(2)$ by using $\underline{\dim}\tau^k S(2) = \Phi^k \underline{\dim} S(2)$. Then the dimension vectors of the modules that are not quasi-simple are determined by the fact that they fit into an Auslander-Reiten sequence and that dimension vectors are additive on short exact sequences. The nice symmetric picture we get is the following, spreading again infinitely to the left, right and downwards:



On the right and left end of this picture, we already observe the exponential behaviour that we proved in the sections before. This finally finishes the example.

3.6 Applications

In this section we study applications of the developed theory to the study of modules. We follow [KS02]. In the appendix, in section B.4 and B.5 we collect some lemmata that are needed to give the proofs. They fit better in the appendix since they work for all finite-dimensional hereditary algebras and not just for wild algebras.

Proposition 3.6.1. *(i) Let U be a nonzero preprojective module and X a nonzero module without indecomposable preinjective direct summands. Then*

$$\lim_{t\to\infty} \frac{1}{\rho^t} \dim \operatorname{Hom}(U, \tau^{-t}X) = \lim_{t\to\infty} \frac{1}{\rho^t} \left\langle \underline{\dim} U, \underline{\dim} \tau^{-t}X \right\rangle > 0.$$

(ii) Let V be a nonzero preinjective module and Y a nonzero module without indecomposable preprojective direct summands. Then

$$\lim_{t\to\infty} \frac{1}{\rho^t} \dim \operatorname{Hom}(\tau^{t}Y, V) = \lim_{t\to\infty} \frac{1}{\rho^t} \left\langle \underline{\dim} \tau^{t}Y, \underline{\dim} V \right\rangle > 0.$$

Proof. We prove (i): Since U is preprojective, we have, using the Auslander-Reiten formula 1.1.2, $\operatorname{Ext}(U, \tau^{-t}X) \cong D\operatorname{Hom}(\tau^{-t}X, \tau U) = 0$ for t big by Lemma B.1.1, proving the first equality. For the second we use Theorem 1.3.2 (i) and get

$$\begin{aligned} \lim_{t\to\infty} \frac{1}{\rho^t} \left\langle \underline{\dim} U, \underline{\dim} \tau^{-t}X \right\rangle &= \left\langle \underline{\dim} U, \lim_{t\to\infty} \frac{1}{\rho^t} \tau^{-t}X \right\rangle \\ &= \left\langle \underline{\dim} U, \lambda_X^- x^- \right\rangle \\ &= \lambda_X^- \left\langle \underline{\dim} U, x^- \right\rangle, \end{aligned}$$

with a $\lambda_X^- > 0$. We can assume without loss of generality that U is indecomposable. Then consider the case that U is not projective. Then we get using Proposition 1.2.2

$$\left\langle \underline{\dim} U, x^- \right\rangle = -\left\langle x^-, \Phi \underline{\dim} U \right\rangle = -\left\langle x^-, \underline{\dim} \tau U \right\rangle > 0,$$

where we used in the last step Theorem 3.3.2. Now if $U = P(i)$ is projective then we get by Lemma 3.3.1:

$$\left\langle \underline{\dim} U, x^- \right\rangle = \left\langle \underline{\dim} P(i), x^- \right\rangle = x_i^- > 0$$

since x^- is strictly positive. In any case, we are done. (ii) is proven dually. □

Proposition 3.6.2. *Assume X is a module without indecomposable preprojective direct summands and Y a module without indecomposable preinjective direct summands. Then there is a natural number N such that for all $t \geq N$ we have*

$$\operatorname{Hom}(X, \tau^{-t} Y) = \operatorname{Hom}(\tau^t X, Y) = 0.$$

Proof. We can write $X = X' \oplus I$ with X' regular and I preinjective and $Y = Y' \oplus P$ with Y' regular and P preprojective. Thus by Lemma 3.4.3 and the fact that τ^- is an equivalence on regular modules there is a natural number N such that for all $t \geq N$ we get

$$\operatorname{Hom}(X', \tau^{-t} Y') = \operatorname{Hom}(\tau^t X', Y') = 0.$$

Thus we get for all $t \geq N$

$$\begin{aligned} \operatorname{Hom}(X, \tau^{-t} Y) &= \operatorname{Hom}(X', \tau^{-t} Y') \oplus \operatorname{Hom}(X', \tau^{-t} P) \oplus \operatorname{Hom}(I, \tau^{-t} Y') \oplus \operatorname{Hom}(I, \tau^{-t} P) \\ &= 0 \oplus 0 \oplus 0 \oplus 0 \\ &= 0, \end{aligned}$$

where we used our former observation and Proposition B.1.2. In the same way we can show $\operatorname{Hom}(\tau^t X, Y) = 0$ for all $t \geq N$, which finishes the proof. □

Lemma 3.6.3. *For regular modules X and Y there is an N such that for all $t \geq N$, we get*

$$\begin{aligned} \left\langle \underline{\dim} X, \underline{\dim} \tau^t Y \right\rangle &= \dim \operatorname{Hom}(X, \tau^t Y) \\ = \left\langle \underline{\dim} \tau^{-t} X, \underline{\dim} Y \right\rangle &= \dim \operatorname{Hom}(\tau^{-t} X, Y). \end{aligned}$$

Proof. By the Auslander-Reiten formula Theorem 1.1.2 we have

$$\operatorname{Ext}(X, \tau^t Y) = D \operatorname{Hom}(\tau^{t-1} Y, X).$$

The latter is zero for t big enough by Lemma 3.4.3. That shows the first equality. The equality of the two Hom-spaces is due to τ being an equivalence on the category of regular modules and the equality of the two expressions on the left is due to the fact that $\langle -, - \rangle$ is Φ-stable for the Coxeter transformation Φ of H and since $\underline{\dim} \tau Z = \Phi \underline{\dim} Z$. □

Lemma 3.6.4. *Let X and Y be modules such that X does not have a nontrivial preinjective direct summand and Y does not have a nontrivial preprojective direct summand. Then there is a natural number N such that for all integers $t \geq N$ we have*

$$\left\langle \underline{\dim} \tau^{-t} X, \underline{\dim} Y \right\rangle = \dim \operatorname{Hom}(\tau^{-t} X, Y).$$

Proof. We need to show that $\mathrm{Ext}(\tau^{-t}X, Y)$ vanishes for t big enough. Write $X = P_X \oplus R_X$ with P_X preprojective and R_X regular and $Y = R_Y \oplus I_Y$ with R_Y regular and I_Y preinjective. Then we get

$$\begin{aligned}\mathrm{Ext}(\tau^{-t}X, Y) = &\mathrm{Ext}(\tau^{-t}P_X, R_Y) \oplus \mathrm{Ext}(\tau^{-t}P_X, I_Y)\\ &\oplus \mathrm{Ext}(\tau^{-t}R_X, R_Y) \oplus \mathrm{Ext}(\tau^{-t}R_X, I_Y).\end{aligned}$$

Now by Proposition 3.6.3 there is an N such that for all $t \geq N$ we have $\mathrm{Ext}(\tau^{-t}R_X, R_Y) = 0$. Now the other three terms disappear because of the Auslander-Reiten formula 1.1.2 and Lemma B.1.2. □

Lemma 3.6.5. *Let Y be a preprojective module. Then for all modules X there is a natural number N such that $\mathrm{Hom}(\tau^{-t}X, Y) = 0$ for all $t \geq 0$.*

Proof. This follows directly from Lemma B.1.1. □

We denote by $\mathcal{S}_p$, $\mathcal{S}_r$ and $\mathcal{S}_i$ sets of representatives of isomorphism classes of simple modules which are preprojective, regular and preinjective, respectively. Let in the rest of this section $\{e, f\}$ be a complete set of orthogonal idempotents such that He is the projective cover of the direct sum of all modules in $\mathcal{S}_p$ and Hf is the projective cover of the modules in $\mathcal{S}_r \cup \mathcal{S}_i$. This is prossible since $H = kQ$ is a basic algebra, i.e. every indecomposable projective appears only once as a direct summand of the algebra.

Theorem 3.6.6. *Let X be a nonzero module without indecomposable preinjective direct summands and z a positive integer. Then there is a natural number N such that for all integers $t \geq N$ the following hold:*

(i) $f\tau^{-t}X$ is a projective fHf-module, $\mathrm{top}\,(f\tau^{-t}X) \cong \mathrm{top}\,(\tau^{-t}X)$ and every simple module from $\mathcal{S}_r \cup \mathcal{S}_i$ appears with multiplicity at least z in $\mathrm{top}\,(\tau^{-t}X)$.

(ii) $e\tau^{-t}X$ is an injective eHe-module , $\mathrm{soc}\,(e\tau^{-t}X) \cong \mathrm{soc}\,(\tau^{-t}X)$ and every simple module from $\mathcal{S}_p$ occurs with multiplicity at least z in $\mathrm{soc}\,(\tau^{-t}X)$.

Proof. We prepare the proof of (i) and (ii): We claim that there is a natural number N such that for all $t \geq N$ all of the following conditions are satisfied simultaneously:

(a) $\mathrm{Ext}\,(\tau^{-t}X, S) \cong D\,\mathrm{Hom}\left(S, \tau^{1-t}X\right) = 0$ for all $S \in \mathcal{S}_r \cup \mathcal{S}_i$.

(b) $\dim \mathrm{Hom}\,(\tau^{-t}X, S) = \left\langle \underline{\dim}\tau^{-t}X, \underline{\dim}S\right\rangle \geq z$ for all $S \in \mathcal{S}_r \cup \mathcal{S}_i$.

(c) $\mathrm{Hom}\,(\tau^{-t}X, S) = 0$ for all $S \in \mathcal{S}_p$.

(d) $\dim \mathrm{Hom}\,(S, \tau^{-t}X) = \left\langle \underline{\dim}S, \underline{\dim}\tau^{-t}X\right\rangle \geq z$ for all $S \in \mathcal{S}_p$.

We do this as follows: As there are only finitely many simple modules (up to isomorphism) and as we are only concerned with the finitely many statements (a), (b), (c) and (d), it is enough to prove the existence of the stated N for every statement and every simple module seperately. We prove (a): The first isomorphism is just the Auslander-Reiten formula Theorem 1.1.2. Since X has no preinjective direct summands and S has no preprojective direct summands, it follows by Proposition 3.6.2 that there is an N such that $\mathrm{Hom}\,(S, \tau^{1-t}X) = 0$ (and therefore also $D\,\mathrm{Hom}\,(S, \tau^{1-t}X) = 0$) for all $t \geq N$.

Now let us prove (b): The first equality is true for large t by Lemma 3.6.4. The second equality can be reached for large t by Theorem 1.3.2 (iii) since $\rho > 1$ by Theorem 1.3.1. This shows (b). (c) follows directly from Lemma 3.6.5.

Next we prove (d): For the first equation we use the Auslander-Reiten formula 1.1.2 and get $\operatorname{Ext}(S, \tau^{-t}X) \cong D\operatorname{Hom}(\tau^{-t}X, \tau S)$, which is zero for large t by Lemma B.1.1. The second formula follows for large t directly from Proposition 3.6.1 (i) since $\rho > 1$ again.

After this preparation, we prove (i): Let $t \geq N$ be a fixed number an look at the minimal projective resolution

$$0 \longrightarrow P_1 \xrightarrow{\alpha} P_0 \xrightarrow{\beta} \tau^{-t}X \longrightarrow 0$$

in $\operatorname{mod}(H)$. Applying $\operatorname{Hom}(-, S)$ for $S \in \mathcal{S}_r \cup \mathcal{S}_i$ and using that $\operatorname{Ext}(P, -) = 0$ for projective P gives an exact sequence

$$0 \longrightarrow \operatorname{Hom}(\tau^{-t}X, S) \xrightarrow{\operatorname{Hom}(\beta,S)} \operatorname{Hom}(P_0, S) \xrightarrow{\operatorname{Hom}(\alpha,S)} \operatorname{Hom}(P_1, S) \xrightarrow{\delta} \operatorname{Ext}(\tau^{-t}X, S) \longrightarrow 0.$$

We have $\operatorname{im}(\alpha) = \ker(\beta)$, which is small since β is a projective cover. Thus $\operatorname{im}(\alpha)$ is contained in $\operatorname{rad}(P_0)$, the sum of all small submodules. But that shows that for any homomorphism $f : P_0 \to S$ we have $0 = f \circ \alpha = \operatorname{Hom}(\alpha, S)(f)$ since $f(\operatorname{im}(\alpha)) \subseteq f(\operatorname{rad}(P_0)) \subseteq \operatorname{rad}(S) = 0$. Thus we have $\operatorname{Hom}(\alpha, S) = 0$. Also, $\operatorname{Ext}(\tau^{-t}X, S) = 0$ by (a). Then exactness shows that $\operatorname{Hom}(P_1, S) = 0$ as well. That $\operatorname{Hom}(P_1, S) = 0$ for all $S \in \mathcal{S}_r \cup \mathcal{S}_i$ shows that $fP_1 = 0$ by Lemma B.5.4. From this and Lemma B.5.5 we conclude $f\tau^{-t}X \cong fP_0$, which is projective by Lemma B.5.1 (i), which proves the first claim in (i). In order to show the second claim in (i), we observe that we have according to Lemma B.5.2 and B.5.3 (i) isomorphisms

$$\begin{aligned}\operatorname{Hom}_H(\operatorname{top}(\tau^{-t}X), S) &\cong \operatorname{Hom}_H(\tau^{-t}X, S)\\ &\cong \operatorname{Hom}_{fHf}(f\tau^{-t}X, fS)\\ &\cong \operatorname{Hom}_{fHf}(\operatorname{top}(f\tau^{-t}X), fS),\end{aligned}$$

for all $S \in \mathcal{S}_r \cup \mathcal{S}_i$. As S (and therefore also fS) is one-dimensional, the k-dimension of these Hom-spaces is precisely the number of times S appears in the top. This shows the relation $[\operatorname{top}(\tau^{-t}X) : S] = [\operatorname{top}(f\tau^{-t}X) : fS]$ for all $S \in \mathcal{S}_r \cup \mathcal{S}_i$ and that those multiplicities are at least z by (b). (c) shows that $[\operatorname{top}(\tau^{-t}X) : S] = 0$ for all $S \in \mathcal{S}_p$. Since the fS with $S \in \mathcal{S}_r \cup \mathcal{S}_i$ are exactly the simple modules in fHf due to Lemma B.5.6, all of this shows indeed that $\operatorname{top}(\tau^{-t}X) \cong \operatorname{top}(f\tau^{-t}X)$ (in the sense that the simple direct summands of these semisimple modules – which are set-theoretically the same since $S = fS$ – match up). This finishes the proof of (i).

Now we prove (ii). We look at a minimal injective resolution

$$0 \longrightarrow \tau^{-t}X \xrightarrow{\alpha} I_0 \xrightarrow{\beta} I_1 \longrightarrow 0$$

of $\tau^{-t}X$ in $\mathrm{mod}(H)$. For each $S \in \mathcal{S}_p$ we apply $\mathrm{Hom}(S, -)$. Using that $\mathrm{Ext}(-, I) = 0$ for injective modules I we get an exact sequence

$$0 \longrightarrow \mathrm{Hom}(S, \tau^{-t}X) \xrightarrow{\mathrm{Hom}(S,\alpha)} \mathrm{Hom}(S, I_0) \xrightarrow{\mathrm{Hom}(S,\beta)} \mathrm{Hom}(S, I_1) \xrightarrow{\delta} \mathrm{Ext}(S, \tau^{-t}X) \longrightarrow 0.$$

We have $\ker(\beta) = \mathrm{im}(\alpha)$, which is large since α is an injective envelope and thus contains $\mathrm{soc}(I_0)$, which is the intersection of all large submodules of I_0. Now if $f : S \to I_0$ is a homomorphism, then $\mathrm{im}(f) \subseteq \mathrm{soc}(I_0) \subseteq \ker(\beta)$, since S is its own socle. We get $\mathrm{Hom}(S, \beta)(f) = \beta \circ f = 0$ and thus $\mathrm{Hom}(S, \beta) = 0$. We also have $\mathrm{Ext}(S, \tau^{-t}X) \cong D\,\mathrm{Hom}(\tau^{-t-1}X, S) = 0$ by the Auslander-Reiten formula Theorem 1.1.2 and (c). Thus by exactness we get $\mathrm{Hom}(S, I_1) = 0$. This shows $eI_1 = 0$ by Lemma B.5.4. By Lemma B.5.5 we conclude that $e\tau^{-t}X \cong eI_0$, which is an injective eHe-module by Lemma B.5.1, proving the first claim in (ii). For the second claim in (ii) we observe that we have by Lemma B.5.2 and B.5.3 (ii) isomorphisms

$$\begin{aligned} \mathrm{Hom}_H(S, \mathrm{soc}(\tau^{-t}X)) &\cong \mathrm{Hom}_H(S, \tau^{-t}X) \\ &\cong \mathrm{Hom}_{eHe}(eS, e\tau^{-t}X) \\ &\cong \mathrm{Hom}_{eHe}(eS, \mathrm{soc}(e\tau^{-t}X)) \end{aligned}$$

for all $S \in \mathcal{S}_p$. As before we get $[\mathrm{soc}(\tau^{-t}X) : S] = [\mathrm{soc}(e\tau^{-t}X) : eS]$ for all $S \in \mathcal{S}_p$ and that those multiplicities are at least z by (d). Furthermore (a) shows $[\mathrm{soc}(\tau^{-t}X) : S] = 0$ for all $S \in \mathcal{S}_r \cup \mathcal{S}_i$. Since eS with $S \in \mathcal{S}_p$ are according to Lemma B.5.6 (ii) precisely the simple eHe-modules, we conclude that $\mathrm{soc}(\tau^{-t}X) \cong \mathrm{soc}(e\tau^{-t}X)$ (again in the sense that their simples match up), which completes the proof. □

Lemma 3.6.7. *Let M be a module.*

(i) For a projective cover $\beta : P \twoheadrightarrow M$ we have $\mathrm{top}(P) \cong \mathrm{top}(M)$.

(ii) For an injective envelope $\alpha : M \hookrightarrow I$ we have $\mathrm{soc}(M) \cong \mathrm{soc}(I)$.

Proof. We have

$$\begin{aligned} \mathrm{top}(M) &\cong \mathrm{top}(P/\ker(\beta)) \\ &\cong [P/\ker(\beta)]\,/\,\mathrm{rad}(P/\ker(\beta)) \\ &= [P/\ker(\beta)]\,/\,[(\mathrm{rad}(P) + \ker(\beta))\,/\,\ker(\beta)] \\ &\cong P/\,[\mathrm{rad}(P) + \ker(\beta)] \\ &\cong P/\,\mathrm{rad}(P) \\ &\cong \mathrm{top}(P), \end{aligned}$$

where we used that $\ker(\beta) \subseteq \mathrm{rad}(P)$, since $\ker(\beta)$ is small and $\mathrm{rad}(P)$ is the sum of all small submodules. This proves (i). For (ii) we compute

$$\mathrm{soc}(M) \cong \mathrm{im}(\alpha) \cap \mathrm{soc}(I) = \mathrm{soc}(I),$$

where we used that $\mathrm{soc}(I) \subseteq \mathrm{im}(\alpha)$ since the latter is large and the former is the intersection of all large submodules. □

For a module M, denote by $P_0(M)$ and $P_1(M)$ the projective modules (determined up to isomorphism) appearing in a minimal projective resolution

$$0 \longrightarrow P_1(M) \longrightarrow P_0(M) \longrightarrow M \longrightarrow 0 .$$

In the same way, donate by $I_0(M)$ and $I_1(M)$ the injective modules appearing in a minimal injective resolution

$$0 \longrightarrow M \longrightarrow I_0(M) \longrightarrow I_1(M) \longrightarrow 0 .$$

Theorem 3.6.8. *Let X and Y be nonzero modules without indecomposable preinjective direct summands. Then the following are equivalent:*

(i) $\underline{\dim} X = \underline{\dim}(Y)$.

(ii) $P_0(\tau^{-m}X) \cong P_0(\tau^{-m}Y)$ *and* $P_1(\tau^{-m}X) \cong P_1(\tau^{-m}Y)$ *for some natural number* m.

(iii) $P_0(\tau^{-t}X) \cong P_0(\tau^{-t}Y)$ *and* $P_1(\tau^{-t}X) \cong P_1(\tau^{-t}Y)$ *for all but finitely many natural numbers* t.

(iv) $I_0(\tau^{-m}X) \cong I_0(\tau^{-m}Y)$ *and* $I_1(\tau^{-m}X) \cong I_1(\tau^{-m}Y)$ *for some natural number* m.

(v) $I_0(\tau^{-t}X) \cong I_0(\tau^{-t}Y)$ *and* $I_1(\tau^{-t}X) \cong I_1(\tau^{-t}Y)$ *for all but finitely many natural numbers* t.

Proof. We will prove $(i) \Rightarrow (iii) \Rightarrow (ii) \Rightarrow (i)$ and $(i) \Rightarrow (v) \Rightarrow (iv) \Rightarrow (i)$. The implications $(iii) \Rightarrow (ii)$ and $(v) \Rightarrow (iv)$ are clear.

We now show $(ii) \Rightarrow (i)$. We have by additivity of dimension vectors on short exact sequences

$$\begin{aligned} \underline{\dim}\tau^{-m}X &= \underline{\dim}P_0(\tau^{-m}X) - \underline{\dim}P_1(\tau^{-m}X) \\ &= \underline{\dim}P_0(\tau^{-m}Y) - \underline{\dim}P_1(\tau^{-m}Y) \\ &= \underline{\dim}\tau^{-m}Y. \end{aligned}$$

From this we get by Proposition 1.1.1 and Lemma 1.1.3 that

$$\begin{aligned} \underline{\dim}X &= \underline{\dim}\tau^{m}\tau^{-m}X \\ &= \Phi^m \underline{\dim}\tau^{-m}X \\ &= \Phi^m \underline{\dim}\tau^{-m}Y \\ &= \underline{\dim}\tau^{m}\tau^{-m}Y \\ &= \underline{\dim}Y, \end{aligned}$$

so we are done. For the proof of $(iv) \Rightarrow (i)$ we do precisely the same, but using the minimal injective resolutions of $\tau^{-m}X$ and $\tau^{-m}Y$ instead.

We now prove the remaining implications $(i) \Rightarrow (iii)$ and $(i) \Rightarrow (v)$. So assume $\underline{\dim}X = \underline{\dim}Y$. From the proof of Theorem 3.6.6 we know that there exists a natural number N such that for all natural numbers $t \geq N$ we have the following:

(a) $\langle \underline{\dim}\tau^{-t}X, \underline{\dim}S \rangle$ is for all $S \in \mathcal{S}_r \cup \mathcal{S}_i$ the multiplicity of S in $\mathrm{top}(\tau^{-t}X)$ and the multiplicity of $S \in \mathcal{S}_p$ in $\mathrm{top}(\tau^{-t}X)$ is zero.

(b) $\langle \underline{\dim}\tau^{-t}Y, \underline{\dim}S\rangle$ is for all $S \in \mathcal{S}_r \cup \mathcal{S}_i$ the multiplicity of S in $\mathrm{top}(\tau^{-t}Y)$ and the multiplicity of $S \in \mathcal{S}_p$ in $\mathrm{top}(\tau^{-t}Y)$ is zero.

(c) $\langle \underline{\dim}S, \underline{\dim}\tau^{-t}X\rangle$ is for all $S \in \mathcal{S}_p$ the multiplicity of S in $\mathrm{soc}(\tau^{-t}X)$ and the multiplicity of $S \in \mathcal{S}_r \cup \mathcal{S}_i$ in $\mathrm{soc}(\tau^{-t}X)$ is zero.

(d) $\langle \underline{\dim}S, \underline{\dim}\tau^{-t}Y\rangle$ is for all $S \in \mathcal{S}_p$ the multiplicity of S in $\mathrm{soc}(\tau^{-t}Y)$ and the multiplicity of $S \in \mathcal{S}_r \cup \mathcal{S}_i$ in $\mathrm{soc}(\tau^{-t}Y)$ is zero.

Let $t \geq N$. We have by Lemma 1.1.3 $\underline{\dim}\tau^{-t}X = \Phi^{-t}\underline{\dim}X = \Phi^{-t}\underline{\dim}Y = \underline{\dim}\tau^{-t}Y$ and thus we get for all $S \in \mathcal{S}_r \cup \mathcal{S}_i$ that $\langle \underline{\dim}\tau^{-t}X, \underline{\dim}S\rangle = \langle \underline{\dim}\tau^{-t}Y, \underline{\dim}S\rangle$ and for all $S \in \mathcal{S}_p$ $\langle \underline{\dim}S, \underline{\dim}\tau^{-t}X\rangle = \langle \underline{\dim}S, \underline{\dim}\tau^{-t}Y\rangle$. Since the semisimple modules $\mathrm{top}(\tau^{-t}X)$ and $\mathrm{top}(\tau^{-t}Y)$ do not contain simple modules from $\mathcal{S}_p$ we conclude $\mathrm{top}(\tau^{-t}X) \cong \mathrm{top}(\tau^{-t}Y)$ and similarly $\mathrm{soc}(\tau^{-t}X) \cong \mathrm{soc}(\tau^{-t}Y)$. This shows $\mathrm{top}(P_0(\tau^{-t}X)) \cong \mathrm{top}(P_0(\tau^{-t}Y))$ and $\mathrm{soc}(I_0(\tau^{-t}X)) \cong \mathrm{soc}(I_0(\tau^{-t}Y))$ by Lemma 3.6.7. Since projective modules are determined by their top and injective modules are determined by their socle we conclude $P_0(\tau^{-t}X) \cong P_0(\tau^{-t}Y)$ and $I_0(\tau^{-t}X) \cong I_0(\tau^{-t}Y)$.

Using additivity of dimension vectors on short exact sequences we get – similarly to what we did in the proof of $(ii) \Rightarrow (i)$ – that $\underline{\dim}P_1(\tau^{-t}X) = \underline{\dim}P_1(\tau^{-t}Y)$ and $\underline{\dim}I_1(\tau^{-t}X) = \underline{\dim}I_1(\tau^{-t}Y)$. We decompose our modules in indecomposable modules: $P_1(\tau^{-t}X) \cong \bigoplus_{i=1}^n P(i)^{r_i^X}$ and $P_1(\tau^{-t}Y) \cong \bigoplus_{i=1}^n P(i)^{r_i^Y}$. Then using additivity of dimension vectors on direct sums we get

$$\sum_{i=1}^n r_i^X \underline{\dim}P(i) = \sum_{i=1}^n r_i^Y \underline{\dim}P(i).$$

Since the dimension vectors of the projective indecomposable modules form a basis (see the Introduction) we conclude $r_i^X = r_i^Y$ for all $i \in \{1, \ldots, n\}$. This shows $P_1(\tau^{-t}X) \cong P_1(\tau^{-t}Y)$. In the same way we show $I_1(\tau^{-t}X) \cong I_1(\tau^{-t}Y)$, which finishes the proof. □

Remark 3.6.9. We want to remark here that there are also dual statements of the two theorems in this section. In the dual statements, He is the projective cover of the direct sum of the simple modules in $\mathcal{S}_p \cup \mathcal{S}_r$ and Hf is the projective cover of the direct sum of the simple modules in $\mathcal{S}_i$. The dual statements can be found in the original article [KS02].

We further remark that there was a minor mistake in the proof of the second theorem in the original article. It stated that $eP_0(\tau^{-t}X) = eP_0(\tau^{-t}Y) = fI_0(\tau^{-t}X) = fI_0(\tau^{-t}Y) = 0$. None of this is correct in general and we thank Otto Kerner for clarifying this.

Appendix A

Terminology of quivers

Let $Q = (Q_0, Q_1, s, t)$ be a quiver, that is Q_0 and Q_1 are finite sets of vertices and arrows and $s, t : Q_1 \to Q_0$ assign a starting vertex (or source) and terminal vertex (or target) to every arrow. Technically, a quiver is just a finite directed graph. For an arrow $\alpha \in Q_1$ with $s(\alpha) = x$ and $t(\alpha) = y$ we write

$$x \xrightarrow{\alpha} y.$$

For $y \in Q_0$, we write y^- for the set of all vertices that are the sources of arrows ending in y and similarly x^+ for the set of vertices that are targets of arrows starting in x.

Definition A.0.1 (sink, source). $y \in Q_0$ is called a *source* if $y^- = \emptyset$ and a *sink* if $y^+ = \emptyset$.

Definition A.0.2 (path). If $\alpha_1, \ldots, \alpha_m$ are arrows in Q with $t(\alpha_i) = s(\alpha_{i+1})$ then the product $\omega = \alpha_m \cdots \alpha_1$ is a *path* which starts in $s(\alpha_1)$ and ends in $t(\alpha_m)$. We write $x \rightsquigarrow y$ if there is a path starting in x and ending in y.

Definition A.0.3 ((Oriented) cycle). A sequence $(i_1, \ldots, i_m)$ of pairwise different vertices in Q_0 is called a *cycle* (of length m) if for all $r \in \{1, \ldots, m\}$ there is an arrow $i_{r-1} \to i_r$ or an arrow $i_r \to i_{r-1}$ (where $i_0 := i_m$).

The cycle is called *oriented* provided there is always an arrow $i_{r-1} \to i_r$.

Definition A.0.4 (Directed quiver). The quiver Q is called *directed* if there is no oriented cycle of length at least 1 in it.

Remark A.0.5. Clearly, every directed quiver does contain at least one source and at least one sink.

Definition A.0.6 (Neighbours). $x, y \in Q_0$ are called *neighbours* if there is an arrow $x \to y$ or an arrow $y \to x$.

Definition A.0.7 (Connected quiver). Q is called *connected* if for any partition $Q_0 = Q_0' \coprod Q_0''$ with Q_0' and Q_0'' nonempty there exists $x \in Q_0'$ and $y \in Q_0''$ such that x and y are neighbours.

It is easy to show that Q is connected if and only if for every two vertices $x, y \in Q_0$ there is a sequence $(x = x_0, \ldots, x_m = y)$ such that for all $i \in \{1, \ldots, m\}$, x_{i-1} and x_i are neighbours.

Appendix B

Facts on hereditary algebras

In this appendix, we collect some knowledge about hereditary algebras. Let throughout H be a finite-dimensional connected hereditary k-algebra. In this appendix we also follow our convention of omitting subscripts, so for example $\mathrm{Ext} = \mathrm{Ext}^1_H$, $\mathrm{Hom} = \mathrm{Hom}_H$, $\dim = \dim_k$, etc.

B.1 Preprojective, regular and preinjective modules

For an indecomposable preprojective module P, denote by $\nu(P)$ the smallest non-negative integer such that $\tau^{\nu(P)}P$ is projective. Let $\mu(I) \geq 0$ be the dual number for an indecomposable preinjective module (i.e. $\tau^{-\mu(I)}I$ is injective). The following fact, taken from [ARS97, ch. VIII Corollary 1.4], is very important:

Lemma B.1.1. *Let Y be an indecomposable preprojective module and let X be indecomposable with $\mathrm{Hom}(X, Y) \neq 0$. Then X is also preprojective and $\nu(X) \leq \nu(Y)$.*

Also the dual statement is correct: Let X be indecomposable preinjective and let Y be indecomposable with $\mathrm{Hom}(X, Y) \neq 0$. Then Y is also indecomposable preinjective and $\mu(Y) \leq \mu(X)$.

From that the following follows easily:

Proposition B.1.2. *Let X preprojective, Y regular and Z preinjective, then*

$$\mathrm{Hom}(Z, X) = \mathrm{Hom}(Z, Y) = \mathrm{Hom}(Y, X) = 0.$$

Corollary B.1.3. *(i) Submodules of regular modules cannot have nontrivial preinjective direct summands.*

(ii) Factor modules of regular modules cannot have nontrivial preprojective direct summands.

(iii) A module which is both a submodule of some regular module and a factor module of some (potentially different) regular module is itself regular.

Proof. Let $0 \to K \to R \to I \to 0$ a short exact sequence with R regular. If $K = K_1 \oplus K_2$ with K_1 nonzero and preinjective, then $\mathrm{Hom}(K_1, R)$ contains the nontrivial injection $K_1 \to K \to R$, which cannot be true.

If $I = I_1 \oplus I_2$ with I_1 nonzero and preprojective, then $\mathrm{Hom}(R, I_1)$ contains the nontrivial surjection $R \to I \to I_1$, which also cannot be true. □

Lemma B.1.4. *Let*

$$0 \longrightarrow U \xrightarrow{\alpha} V \xrightarrow{\beta} W \longrightarrow 0$$

be a short exact sequence. Then we get:

(i) If U and W do not have indecomposable preprojective direct summands, then also V does not have an indecomposable preprojective direct summand.

(ii) If U and W do not have indecomposable preinjective direct summands, then also V does not have an indecomposable preinjective direct summand.

(iii) If U and W are regular, then so is V.

Proof. (iii) follows immediately from (i) and (ii).

Assume U and W do not have an indecomposable preprojective direct summand and write $V = V' \oplus P$ with P preprojective. We want to show $P = 0$. By Proposition B.1.2 we have $\mathrm{Hom}(U, P) = 0$, i.e. the map $\pi \circ \alpha$ in the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & U & \xrightarrow{\alpha} & V & \xrightarrow{\beta} & W & \longrightarrow & 0 \\ & & & \searrow^{0} & \downarrow{\scriptstyle \pi} & & & & \\ & & & & P & & & & \end{array}$$

vanishes. We claim $W = \beta(V') \oplus \beta(P)$. $W = \beta(V') + \beta(P)$ follows since β is surjective. Let $w \in \beta(V') \cap \beta(P)$, i.e. there are $v' \in V'$ and $p \in P$ such that $w = \beta(v') = \beta(p)$. Then $p - v' \in \ker(\beta) = \mathrm{im}(\alpha)$, so there is some $u \in U$ such that $\alpha(u) = p - v'$. It follows $0 = (\pi \circ \alpha)(u) = \pi(p - v') = p$ and thus $w = \beta(p) = \beta(0) = 0$, proving the claim. Now β is injective on P since $\pi \circ \alpha = 0$. Thus $P \cong \beta(P)$ and so $W \cong \beta(V') \oplus P$ which shows by assumption on W that $P = 0$. Thus we proved (i).

Now assume U and W do not have an indecomposable preinjective direct summand and write $V = V' \oplus I$ with I preinjective. We want to show that $I = 0$. By Proposition B.1.2 we have $\mathrm{Hom}(I, W) = 0$ and so $\beta(I) = 0$. Thus we have $I \subseteq \mathrm{im}(\alpha)$. We claim that $U = \alpha^{-1}(V') \oplus \alpha^{-1}(I)$. That the intersection $\alpha^{-1}(V') \cap \alpha^{-1}(I)$ is zero follows from α being injective. Now let $u \in U$. Then $\alpha(u) = v' + i$ with $v' \in V'$ and $i \in I$. There is some $i' \in \alpha^{-1}(I)$ such that $\alpha(i') = i$ since $I \subseteq \mathrm{im}(\alpha)$. Then $\alpha(u - i') = v' + i - i = v' \in V'$ and thus $u - i' \in \alpha^{-1}(V')$. This shows $u = (u - i') + i' \in \alpha^{-1}(V') + \alpha^{-1}(I')$ and thus the claim. Since $\alpha^{-1}(I) \cong \alpha(\alpha^{-1} I) = I$ it follows that $U \cong \alpha^{-1}(V') \oplus I$ and therefore by assumption on U that $I = 0$. Therefore we proved (ii). □

B.2 Elementary modules

We follow [Ker96] in this section.

Definition B.2.1 (Elementary module). A regular module $E \neq 0$ is called *elementary* if there is no short exact sequence $0 \to U \to E \to V \to 0$ with U, V both nonzero and regular.

Proposition B.2.2. *(i) If E is elementary, then all translates $\tau^m E$ for $m \in \mathbb{Z}$ are as well.*

(ii) Every nonzero regular module M has a filtration

$$M = M_0 \supset M_1 \supset \cdots \supset M_r \supset M_{r+1} = 0$$

such that all factor modules M_i/M_{i+1} are elementary for $i = 0, \ldots, r$. In this situation, we get for all $m \in \mathbb{Z}$ a corresponding filtration

$$\tau^m M = \tau^m M_0 \supset \tau^m M_1 \supset \cdots \supset \tau^m M_r \supset \tau^m M_{r+1} = 0,$$

with elementary factors.

Proof. We prove (i): Assume $\tau^m E$ is not elementary. We get a short exact sequence

$$0 \to U \to \tau^m E \to V \to 0$$

with U and V regular and nonzero. By applying τ^{-m} to this sequence and using Proposition 1.1.1 we thus get a short exact sequence

$$0 \to \tau^{-m} U \to E \to \tau^{-m} V \to 0$$

with $\tau^{-m}U$ and $\tau^{-m}V$ also nonzero regular, contradicting that E is elementary.

Now we prove (ii) using induction on the length of M: length 1 is clear, so let the length be greater than 1.

If M is already elementary, then we are done because of the filtration $M \supset 0$. If M is not elementary, we get a short exact sequence $0 \to U \to M \to V \to 0$ with U and V both nonzero regular. Thus we get a filtration $M \supset U \supset 0$ with factors V and U. V and U have both smaller length than M and thus by induction hypotheses both have filtrations $U = U_0 \supset \cdots \supset U_{r'+1} = 0$ and $V = V_0 \supset \cdots \supset V_{r''+1} = 0$ with elementary factors. Then the preimages V_i' of V_i under the projection $M \to V$ together with the U_i form the desired filtration of M with elementary factors. Here we used that the V_i' sit in a short exact sequence $0 \to U \to V_i' \to V_i \to 0$ with U and V_i regular, which means that V_i' is also regular by Lemma B.1.4.

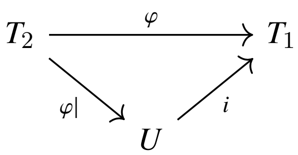
Now let $M = M_0 \supset \cdots \supset M_{r+1} = 0$ be a filtration of regular modules with elementary factors. Then since τ^m is exact on regular modules, the filtration is preserved and we get a filtration of regular modules $\tau^m M = \tau^m M_0 \supset \cdots \supset \tau^m M_{r+1} = 0$. Also by exactness, we get $\tau^m M_i / \tau^m M_{i+1} = \tau^m (M_i/M_{i+1})$ which is also elementary by (i). □

B.3 Some tilting theory

We follow [HR82, ch. 4] in this section.

Lemma B.3.1. *Let T_1, T_2 be indecomposable modules such that* $\operatorname{Ext}(T_1, T_2) = 0$. *Then every morphism $0 \neq \varphi : T_2 \to T_1$ is an epimorphism or a monomorphism.*

Proof. Let $\varphi : T_2 \to T_1$ be nonzero and assume that it is neither an epimorphism nor a monomorphism. Let $U := \operatorname{im}(\varphi)$. Then we get a commutative diagram



where $\varphi|$ is a proper epimorphism and i is a proper monomorphism. Since $\varphi|$ is surjective and $\operatorname{Ext}(T_1/U, -)$ is right exact (we work over a hereditary algebra), also the map $\operatorname{Ext}(T_1/U, \varphi|) : \operatorname{Ext}(T_1/U, T_2) \to \operatorname{Ext}(T_1/U, U)$ is surjective. Thus the short exact sequence $0 \to U \to T_1 \to T_1/U$ (which is an element of $\operatorname{Ext}(T_1/U, U)$) is in the image, i.e. there is a module V and a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T_2 & \xrightarrow{i'} & V & \longrightarrow & T_1/U & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \varphi|} & & \downarrow{\scriptstyle g} & & \| & & \\ 0 & \longrightarrow & U & \xrightarrow{i} & T_1 & \longrightarrow & T_1/U & \longrightarrow & 0, \end{array}$$

where the left square is a pushout. Therefore, the following sequence is exact (where everything except injectivity of the first map is due to the pushout property. The injectivity of the first map follows since i' is injective):

$$0 \longrightarrow T_2 \xrightarrow{\binom{\varphi|}{-i'}} U \oplus V \xrightarrow{(i,g)} T_1 \longrightarrow 0$$

If this sequence would split we would get that $T_1 \oplus T_2 \cong U \oplus V$. We have $U \neq 0 \neq V$ and so by Krull-Remak-Schmidt Theorem, and since T_1 and T_2 are indecomposable, we conclude $U \cong T_1$ or $U \cong T_2$. But this contradicts the fact that $\varphi| : T_2 \to U$ is a proper epimorphism and $i : U \to T_1$ is a proper monomorphism and thus the sequence does not split. But this means precisely that we found a nonzero element in $\operatorname{Ext}(T_1, T_2)$, finishing the proof. □

Corollary B.3.2. *Let T be a module with $\operatorname{Ext}(T, T) = 0$. Then for all oriented cycles*

$$T_1 \to T_2 \to \cdots \to T_n \to T_1$$

of nonzero maps between indecomposable summands T_i of T we get that all maps are isomorphisms.

Proof. Let T_i, T_j and T_k be indecomposable direct summands of T and let

$$T_i \xrightarrow{\varphi} T_j \xrightarrow{\psi} T_k$$

be nonzero maps. By Lemma B.3.1 we get that φ is a monomorphism or an epimorphism and ψ is a monomorphism or an epimorphism. We claim that one of the following assertions is satisfied:

(i) φ and ψ are both monomorphisms.

(ii) φ and ψ are both epimorphisms.

(iii) φ is a monomorphism and ψ is an epimorphism.

Indeed, assume that none of the assertions is true. Then φ is a epimorphism (but no monomorphism) and ψ is a monomorphism (but no epimorphism). But that means that $0 \neq \psi \circ \varphi : T_i \to T_k$ is neither injective, nor surjective, contradicting Lemma B.3.1.

Now assume that we have a chain

$$T_1 \to T_2 \to \cdots \to T_n \to T_1$$

where no map is zero. Then we know by Lemma B.3.1 that all the maps are monomorphisms or epimorphisms. If there were both proper monomorphisms and proper epimorphisms in the chain, then since it is an oriented cycle there would (after deleting isomorphisms in between) appear a part $T_i \to T_j \to T_k$ in it where the first map is a proper epimorphism and the second map is a proper monomorphism. But this contradicts what we showed above. Thus, all maps a monomorphisms or all maps are epimorphisms. Since we work in the finite-dimensional setting, this means that all maps are isomorphisms. □

Let n be the number of indecomposable projective modules in a decomposition of the algebra H into direct summands.

Proposition B.3.3. *Let $T_1, \ldots, T_s$ be pairwise non-isomorphic indecomposable modules and $T = \bigoplus_{i=1}^{s} T_i$. If $\operatorname{Ext}(T,T) = 0$, then the vectors $\{\underline{\dim} T_i\}_{i \in \{1,\ldots,s\}}$ are linearly independent in $\mathbb{Z}^n$. In particular, s cannot be bigger than n.*

Proof. Let Δ be the quiver with vertex set $\{1, \ldots, s\}$ and an arrow $i \to j$ if and only if $i \neq j$ and $\operatorname{Hom}(T_i, T_j) \neq 0$. Then Corollary B.3.2 can be interpreted in the following way: The quiver Δ has no oriented cycles of length ≥ 1. Thus this quiver has a sink, and we rename the modules without loss of generality so that 1 is a sink. Let 2 be a sink of the remaining quiver after removing 1 and so on. This means that our modules are ordered in such a way that $\operatorname{Hom}(T_i, T_j) = 0$ whenever $i < j$.

We look at the linear map

$$d : \mathbb{Z}^n \to \mathbb{Z}^s,\ x \mapsto \left(\langle \underline{\dim} T_1, x\rangle, \ldots, \langle \underline{\dim} T_s, x\rangle\right),$$

where $\langle -, - \rangle$ denotes the homological bilinear form. Then we claim that the images $d\left(\underline{\dim} T_i\right)$ for $i \in \{1, \ldots, s\}$ are linearly independent in $\mathbb{Z}^s$ (which clearly shows that $\underline{\dim} T_1, \ldots, \underline{\dim} T_s$ are linearly independent in $\mathbb{Z}^n$). We write these vectors from now on as column vectors. Then we need to show that the matrix given by

$$M = \left(d\left(\underline{\dim} T_1\right), \ldots, d\left(\underline{\dim} T_s\right)\right) : \mathbb{Z}^s \to \mathbb{Z}^s$$

is a monomorphism. We have $M_{ij} = d\left(\underline{\dim} T_j\right)_i = \langle \underline{\dim} T_i, \underline{\dim} T_j \rangle = \dim \operatorname{Hom}(T_i, T_j) - \dim \operatorname{Ext}(T_i, T_j) = \dim \operatorname{Hom}(T_i, T_j)$ since $\operatorname{Ext}(T,T) = 0$. Thus for $i < j$ we have $M_{ij} = 0$ by construction and for $i = j$ we have $M_{ii} = \dim \operatorname{End}(T_i) \neq 0$. Thus, M is a lower triangular matrix with nonzero diagonal entries and thus injective, proving the claim. □

B.4 A splitting of the module category by idempotents

Let $\{e, f\}$ be a complete set of two orthogonal idempotents in H, i.e. $e^2 = e$, $f^2 = f$, $ef = fe = 0$ and $f + e = 1$. In this section we investigate how to study an H-module M by studying the eHe-module eM and the fHf-module fM seperately.

Definition B.4.1 (Category $\operatorname{mod}(H)_{e,f}$). We define the *category* $\operatorname{mod}(H)_{e,f}$ as follows: Objects are quadruples $(M_e, M_f, \phi_e, \phi_f)$ such that:

(i) M_e is an eHe-module and M_f is an fHf-module.

(ii) $\phi_e : eHf \otimes_{fHf} M_f \to M_e$ is eHe-linear and $\phi_f : fHe \otimes_{eHe} M_e \to M_f$ is fHf-linear

(iii) We have $\phi_e(eaf \otimes \phi_f(fa'e \otimes m_e)) = (eafa'e)m_e$ and $\phi_f(fa'e \otimes \phi_e(eaf \otimes m_f)) = (fa'eaf)m_f$ for all $a, a' \in H$, $m_e \in M_e$ and $m_f \in M_f$.

Morphisms from $(M_e, M_f, \phi_e, \phi_f)$ to $(M'_e, M'_f, \phi'_e, \phi'_f)$ are tuples (g_e, g_f) such that

(i) $g_e : M_e \to M'_e$ is eHe-linear and $g_f : M_f \to M'_f$ is fHf-linear

(ii) The following two diagrams commute:

$$\begin{array}{ccc} eHf \otimes_{fHf} M_f & \xrightarrow{\phi_e} & M_e \\ {\scriptstyle \mathrm{id} \otimes g_f}\downarrow & & \downarrow{\scriptstyle g_e} \\ eHf \otimes_{fHf} M'_f & \xrightarrow{\phi'_e} & M'_e \end{array} \qquad \begin{array}{ccc} fHe \otimes_{eHe} M_e & \xrightarrow{\phi_f} & M_f \\ {\scriptstyle \mathrm{id} \otimes g_e}\downarrow & & \downarrow{\scriptstyle g_f} \\ fHe \otimes_{eHe} M'_e & \xrightarrow{\phi'_f} & M'_f \end{array}$$

Proposition B.4.2. *There are mutually quasi-inverse functors*

$$\operatorname{mod}(H) \underset{\Psi}{\overset{\Phi}{\rightleftarrows}} \operatorname{mod}(H)_{e,f}$$

i.e. the categories $\operatorname{mod}(H)$ *and* $\operatorname{mod}(H)_{e,f}$ *are equivalent.*

Proof. We define Φ on objects as follows: $\Phi(M) = (eM, fM, \phi_e, \phi_f)$, where ϕ_e and ϕ_f are given by scalar multiplication (e.g. $\phi_e(eaf \otimes fm) := eafm$ for $a \in H$, $m \in M$). We define Φ on morphisms as follows: If $g : M \to M'$ is H-linear, then we set $\Phi(g) = \left(g|_e, g|_f\right)$, where the two maps are just the restrictions, i.e. $g|_e : eM \to eM',\ em \mapsto g(em) = eg(m)$.

We now define Ψ on objects: when $(M_e, M_f, \phi_e, \phi_f)$ is in $\operatorname{mod}(H)_{e,f}$, we set

$$\Psi(M_e, M_f, \phi_e, \phi_f) := M_e \oplus M_f$$

with the H-action

$$a(m_e, m_f) := \left(eaem_e + \phi_e(eaf \otimes m_f), fafm_f + \phi_f(fae \otimes m_e)\right).$$

Furthermore, Ψ is defined on morphisms $(g_e, g_f) : (M_e, M_f, \phi_e, \phi_f) \to (M'_e, M'_f, \phi'_e, \phi'_f)$ by setting $\Psi(g_e, g_f) := g_e \oplus g_f : M_e \oplus M_f \to M'_e \oplus M'_f$, $(m_e, m_f) \mapsto (g_e(m_e), g_f(m_f))$.

One now can show in straightforward computations that this indeed makes Φ and Ψ two well-defined functors.

In order to show that they are mutually quasi-inverse, we have to construct natural isomorphisms $\eta : \mathrm{Id}_{\operatorname{mod}(H)_{e,f}} \Rightarrow \Phi \circ \Psi$ and $\epsilon : \Psi \circ \Phi \Rightarrow \mathrm{Id}_{\operatorname{mod}(H)}$:

Let $(M_e, M_f, \phi_e, \phi_f) \in \operatorname{mod}(H)_{e,f}$. Then we define $\eta_e : M_e \to e(M_e \oplus M_f) = M_e \oplus 0,\ m_e \mapsto (m_e, 0)$ and $\eta_f : M_f \to f(M_e \oplus M_f) = 0 \oplus M_f,\ m_f \mapsto (0, m_f)$. This defines a natural isomorphism $\eta_{(M_e, M_f, \phi_e, \phi_f)} := \left(\eta_e, \eta_f\right) : (M_e, M_f, \phi_e, \phi_f) \to (\Phi \circ \Psi)(M_e, M_f, \phi_e, \phi_f)$.

Let now $M \in \text{mod}(H)$. Then we define $\epsilon_M : (\Psi \circ \Phi)(M) = eM \oplus fM \to M$ by $(em, fm') \mapsto em + fm'$. Since we refer to this specific statement later, we prove in detail that this is an isomorphism: We first need to show that ϵ_M is H-linear: Indeed, given $a \in H$ we get

$$\begin{aligned}\epsilon_M(a(em, fm')) &= \epsilon_M(eaem + eafm', fafm' + faem)\\ &= eaem + eafm' + fafm' + faem\\ &= aem + afm'\\ &= a(em + fm')\\ &= a \cdot \epsilon_M(em, fm').\end{aligned}$$

We now show injectivity of ϵ_M: If $\epsilon_M(em, fm') = 0$, then also $em = eem = eem + efm' = e(em + fm') = e \cdot \epsilon_M(em, fm') = 0$ since e and f are orthogonal. Similarly $fm' = 0$, which shows $(em, fm') = 0$, so ϵ_M is injective. For surjectivity just observe that $m = 1m = em + fm = \epsilon_M(em, fm)$.

Naturality of ϵ_M is clear, and so the proof is complete. □

Corollary B.4.3. *An H-module M is determined up to isomorphism by the eHe-module eM, the fHf-module fM and the scalar multiplications*

$$eHf \otimes_{fHf} fM \to eM, \qquad fHe \otimes_{eHe} eM \to fM.$$

Proof. This is a direct consequence of the following statement from the proof of Proposition B.4.2: $\epsilon_M : eM \oplus fM \to M$ is an isomorphism of H-modules. □

In the remainder of this section we follow [KS02]. We thank Otto Kerner for answering questions about the topic and providing the statement and proof of the following lemma.

Lemma B.4.4. *Let $P \to S$ be the projective cover of the preprojective simple module S. Then the following holds:*

(i) If $X \subseteq \text{rad}(P)$ then P/X is preprojective.

(ii) Every composition factor of P is preprojective.

(iii) If $Q \to T$ is the projective cover of a non-preprojective simple module T, then $\text{Hom}(Q, P) = 0$.

Proof. We prove (i): P is local with unique maximal submodule $\text{rad}(P)$. It follows that P/X is local with unique maximal submodule $\text{rad}(P)/X$, and since local modules are indecomposable we get that P/X is indecomposable. Clearly, there is a canonical nonzero map $P/X \to P/\text{rad}(P) = S$, so $\text{Hom}(P/X, S) \neq 0$. By Lemma B.1.2 and since S is preprojective it follows that also P/X is preprojective.

We now prove (ii): Let S' be a composition factor of P. Then $S' = Y/X$ for submodules $X \subset Y \subseteq P$. It follows that $X \subseteq \text{rad}(P)$, so by (i), P/X is preprojective. Clearly, $\text{Hom}(S', P/X) \neq 0$ (just take the inclusion $S' = Y/X \hookrightarrow P/X$), and so again by Lemma B.1.2 we get that S' is preprojective.

Finally, we prove (iii): If the Hom-space was nonzero, we would get that T is a composition factor of P, and so by (ii), T would also be preprojective, a contradiction. □

We denote by $\mathcal{S}_p$, $\mathcal{S}_r$ and $\mathcal{S}_i$ sets of representatives of isomorphism classes of simple modules which are preprojective, regular and preinjective, respectively. Assume from now on that H is a basic hereditary algebra, i.e. every indecomposable projective module appears only once as a direct sum of H. Let in the rest of this and the next section $\{e, f\}$ be a complete set of orthogonal idempotents such that He is the projective cover of the direct sum of all modules in $\mathcal{S}_p$ and Hf is the projective cover of the modules in $\mathcal{S}_r \cup \mathcal{S}_i$.

Corollary B.4.5. *We have* $fHe \cong \mathrm{Hom}(Hf, He) = 0$.

Proof. Since Hom is additive on both summands, we only need to show the statement in the case that e and f are orthogonal primitive idempotents such that $He/\operatorname{rad}(He)$ is a preprojective simple module and $Hf/\operatorname{rad}(Hf)$ is a regular or preinjective simple module. But then the statement follows directly from Lemma B.4.4 (*iii*). □

Corollary B.4.6. *M is determined by the eHe-module eM, the fHf-module fM and the eHe-linear scalar multiplication $eHf \otimes_{fHf} fM \to eM$.*

Proof. This follows directly from the Corollaries B.4.3 and B.4.5. □

The preceding corollary hopefully motivates sufficiently that we study fM and eM seperately in order to gain insides for the module M.

B.5 Interplay between idempotents and modules

Let still H be a basic hereditary algebra and the notation as before.

Lemma B.5.1. *With e and f as above, we have the following:*

(i) If P is a projective H-module, then fP is a projective fHf-module.

(ii) If I is an injective H-module, then eI is an injective eHe-module.

Proof. We first prove (i): Since direct sums of projective modules are projective, we can reduce to the case where P is indecomposable projective. We can write $e = \sum_{i=1}^{r} e_i$ and $f = \sum_{i=r+1}^{n} e_i$ with pairwise orthogonal primitive idempotents e_i. Then $P \cong He_i$ for some i. In case that $i \in \{1, \ldots, r\}$ we get $fP = fHe_i = fHe_ie \subseteq fHe = 0$ by Corollary B.4.5, so fP is projective. In case $i \in \{r+1, \ldots, n\}$ we get $fP = fHe_i = (fHf)(fe_if)$ since $e_i = fe_if = f(fe_if)$. fe_if is an idempotent in fHf and so fP is projective as an fHf-module, which proves (i).

Now we prove (ii): We can again reduce to the case where I is indecomposable injective, thus $I = D(e_iH)$ for some i. In case $i \in \{r+1, \ldots, n\}$ we claim $eI = eD(e_iH) = 0$ (which is an injective module): Let $\varphi : e_iH \to k$ be in $D(e_iH)$. Then $(e\varphi)(e_ia) = \varphi(e_iae) = \varphi(0) = 0$ since $e_iae = fe_iae \subseteq fHe = 0$ by Corollary B.4.5. Thus $e\varphi = 0$ and so $eI = 0$ as claimed. In case $i \in \{1, \ldots, r\}$ we claim that the map

$$\psi : eI = eD(e_iH) \to D\left((ee_ie)(eHe)\right),\ ef \mapsto (x \mapsto f(x))$$

is a well-defined isomorphism of eHe-modules. After that we are done since ee_ie is an idempotent and thus the right term is an injective eHe-module.

ψ is an eHe-module homomorphism since

$$\psi((eae)(eg))(x) = \psi(e(aeeg))(x) = (aeeg)(x)$$

and

$$((eae)\psi(eg))(x) = \psi(eg)(x(eae)) = g(x(eae)) = g(x(aee)) = (aeeg)(x),$$

proving the claim.

For $g, g' \in I$ we have $eg = eg'$ if and only if $(eg)(e_i a) = (eg')(e_i a)$ for all $a \in H$, which is the case if and only if $g((ee_i e)(eae)) = g'((ee_i e)(eae))$ for all $a \in H$. This is equivalent to $\psi(g) = \psi(g')$. Thus we have well-definedness and injectivity of ψ. For surjectivity let $g' \in D((ee_i e)(eHe))$. We define $g \in I$ by setting $g(e_i a) := g'((ee_i e)(eae))$. Then we get $\psi(eg)((ee_i e)(eae)) = g((ee_i e)(eae)) = g(e_i ae) = g'((ee_i e)(eaee)) = g'((ee_i e)(eae))$ and thus $\psi(eg) = g'$, which shows surjectivity of ψ. □

Lemma B.5.2. *(i) Let $S \in \mathcal{S}_r \cup \mathcal{S}_i$. Then $eS = 0$ and fS is a simple fHf-module.*

(ii) Let $S \in \mathcal{S}_p$. Then $fS = 0$ and eS is a simple eHe-module.

Proof. We prove (ii): Let $S \in \mathcal{S}_p$, i.e. $S \cong He_i/\operatorname{rad}(He_i)$ with $i \in \{1, \dots, r\}$ with the notation from the proof of the preceding Lemma. In order to show $fS = 0$ it suffices to show that $e_j S = 0$ for all $j \in \{r+1, \dots, n\}$, which is equivalent to $e_j He_i \subseteq \operatorname{rad}(He_i)$. We have $e_j He_i \cong \operatorname{Hom}(He_j, He_i)$, and this Hom-space does not contain a surjection since $He_j \not\cong He_i$ and both modules are indecomposable projective. All homomorphisms $He_j \to He_i$ are given by right multiplication with an element $e_j a e_i$. That such a homomorphism is not surjective means that $He_j a e_i \subseteq \operatorname{rad}(He_i)$, since $\operatorname{rad}(He_i)$ is the unique maximal submodule of He_i. Thus we get $He_j He_i = \bigcup_{a \in H} He_j a e_i \subseteq \operatorname{rad}(He_i)$ and thus $e_j He_i \subseteq \operatorname{rad}(He_i)$. Consequently we have in fact $fS = 0$.

In order to show that eS is a simple eHe-module we let $0 \subseteq M \subseteq eS$ be an eHe-submodule of eS and aim to prove that $M = 0$ or $M = eS$. We have $M \subseteq S$ as well. We claim that M is an H-submodule of S: Let $a \in H$ and $m \in M$. Then we get

$$\begin{aligned} am &= (eae + eaf + fae + faf)m \\ &= (eae)m + ea(fm) + f(aem) + fa(fm) \\ &= (eae)m + 0 + 0 + 0 \\ &= (eae)m \in M, \end{aligned}$$

where we used that $fS = 0$. Thus $M \subseteq S$ is an H-submodule, and since S is simple we get $M = 0$ or $M = S = eS$, proving that eS is simple. This proves (ii). (i) is proven essentially in the same way. □

Lemma B.5.3. *Let M and N be H-modules. Then we get the following:*

(i) If $eN = 0$, then the restriction map $r_f : \operatorname{Hom}_H(M, N) \to \operatorname{Hom}_{fHf}(fM, fN)$, $g \mapsto g|_f$ is an isomorphism of k-vectorspaces.

(ii) If $fM = 0$, then the restriction map $r_e : \operatorname{Hom}_H(M, N) \to \operatorname{Hom}_{eHe}(eM, eN)$, $g \mapsto g|_e$ is an isomorphism of k-vectorspaces.

Proof. We prove (i): For injectivity, assume that $g : M \to N$ is a homomorphism of H-modules such that $g|_f = 0$, i.e. $g(fm) = 0$ for all $m \in M$. Let $m \in M$. Then

$$g(m) = g(fm + em) = g(fm) + eg(m) = 0 + 0$$

since $eg(m) \in eN = 0$. For surjectivity, let $g' : fM \to fN$ be a homomorphism of fHf-modules. We define $g : M \to N$ by $g(m) := g'(fm)$, which again is clearly a preimage of g' under r_f provided g is really H-linear. In order to prove this, let $a \in H$ and $m \in M$. Then we get

$$g(am) = g'(fam) = g'(fafm + faem) = g'(fafm) = fafg'(fm) = fafg(m),$$

where we used that $fHe = 0$ according to Corollary B.4.5. So we are done if we can show $fafg(m) = ag(m)$. For this it suffices to show $eaeg(m) = eafg(m) = faeg(m) = 0$. Indeed this is the case since $g(m) = g'(fm) \in fN$, $ef = 0$ and $eN = 0$. This proves (i).

Now we prove (ii): Since $fM = 0$ we have $M = eM$, so r_e is only a corestriction and thus clearly injective. For surjectivity, let $g' : eM \to eN$ be eHe-linear. We define $g : M \to N$ by setting $g(m) := g'(em)$. Then clearly $r_e(g) = g'$, provided that g is indeed H-linear: Let $a \in H$ and $m \in M$. Then we get

$$g(am) = g'(eam) = g'(ea(e + f)m) = g'(eaem) = eaeg'(em) = eaeg(m),$$

where we used that $fm = 0$ since $fM = 0$. Then we are done if we can show $eaeg(m) = ag(m)$, which requires showing $eafg(m) = faeg(m) = fafg(m) = 0$. All of this is indeed the case since $fae = 0$ by Corollary B.4.5 and since $g(m) = g'(em) \in eN$ and $fe = 0$. This finishes the proof. □

Lemma B.5.4. *Let P be a projective module and I an injective module. Then we have the following:*

(i) If $\operatorname{Hom}(P, S) = 0$ for all $S \in \mathcal{S}_r \cup \mathcal{S}_i$, then $fP = 0$.

(ii) If $\operatorname{Hom}(S, I) = 0$ for all $S \in \mathcal{S}_p$, then $eI = 0$.

Proof. We can assume that P is indecomposable. Then in the notation of the proof of Lemma B.5.1 we have $P \cong He_i$ for some $i \in \{1, \dots, r\}$. But then $fP = fHe_i = fHe_ie \subseteq fHe = 0$ by Corollary B.4.5. That proves (i).

For (ii) we can also assume that I is indecomposable and thus $I \cong D(e_iH)$ for some $i \in \{1, \dots, n\}$. For $j \in \{1, \dots, r\}$ we have

$$\operatorname{Hom}\left(D\left(e_jH/\operatorname{rad}(e_jH)\right), D(e_iH)\right) \cong \operatorname{Hom}\left(He_j/\operatorname{rad}(He_j), D(e_iH)\right) = 0$$

by assumption, and thus we get $i \in \{r+1, \dots, n\}$. Now let $\varphi : e_iH \to k$ be in $D(e_iH) = I$. Then we get $(e\varphi)(e_ia) = \varphi(e_iae) = 0$, since $e_iae \in e_iHe = fe_iHe \subseteq fHe = 0$ again by Corollary B.4.5. This proves the claim. □

Lemma B.5.5. *Let*

$$0 \longrightarrow X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \longrightarrow 0$$

be a short exact sequence in $\operatorname{mod}(H)$. *Then for any idempotent e, the sequence*

$$0 \longrightarrow eX \xrightarrow{\alpha|_e} eY \xrightarrow{\beta|_e} eZ \longrightarrow 0$$

with the restricted maps is short exact in $\operatorname{mod}(eHe)$.

Proof. Injectivity of $\alpha|_e$ is clear, as well as $\beta|_e \circ \alpha|_e = 0$. Let $ey \in \ker(\beta|_e)$. Then $ey \in \ker(\beta)$ as well, so there is $x \in X$ such that $\alpha(x) = ey$. We conclude

$$\alpha|_e(ex) = \alpha(ex) = e\alpha(x) = eey = ey$$

since e is an idempotent. This proves $\ker(\beta|_e) \subseteq \operatorname{im}(\alpha|_e)$.

Now let $ez \in eZ$. Then by surjectivity of β there is $y \in Y$ such that $\beta(y) = ez$. By the same reasoning as before we get $(\beta|_e)(ey) = ez$, which proves surjectivity of $\beta|_e$. □

Let from now on e and f again as before, i.e. He covers the simple modules in $\mathcal{S}_p$ and Hf covers the simple modules in $\mathcal{S}_r \cup \mathcal{S}_i$.

Lemma B.5.6. *(i) The simple modules of fHf are up to isomorphism precisely the modules fS where $S \in \mathcal{S}_r \cup \mathcal{S}_i$. For $S,\ S' \in \mathcal{S}_r \cup \mathcal{S}_i$ we have $S \cong S'$ as H-modules if and only of $fS \cong fS'$ as fHf-modules.*

(ii) The simple modules of eHe are up to isomorphism precisely the modules eS where $S \in \mathcal{S}_p$. For $S,\ S' \in \mathcal{S}_p$ we have $S \cong S'$ as H-modules if and only of $eS \cong eS'$ as eHe-modules.

Proof. We only prove (i), since (ii) is analogous. The elements fe_if for $i \in \{r+1, \dots, n\}$ clearly constitute a complete set of orthogonal idempotents in fHf. They are also primitive since $(fe_if)(eHf)(fe_if) = e_iHe_i$ and since e_i is a primitive idempotent in H. Therefore, the modules $(fHf)(fe_if)/\operatorname{rad}((fHf)(fe_if))$ are precisely the simple fHf-modules. Now look at the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & f\operatorname{rad}(He_i) & \longrightarrow & fHe_i & \longrightarrow & f(He_i/\operatorname{rad}(He_i)) & \longrightarrow & 0 \\ & & \| & & \| & & \vdots\downarrow & & \\ 0 & \longrightarrow & \operatorname{rad}(fHe_i) & \longrightarrow & fHe_i & \longrightarrow & (fHe_i)/\operatorname{rad}(fHe_i) & \longrightarrow & 0, \end{array}$$

where the upper sequence is induced from the sequence $0 \to \operatorname{rad}(He_i) \to He_i \to He_i/\operatorname{rad}(He_i) \to 0$ by restriction, which gives a short exact sequence by Lemma B.5.5. The lower sequence is the canonical projective resolution of

$$(fHe_i)/\operatorname{rad}(fHe_i) = (fHf)(fe_if)/\operatorname{rad}((fHf)(fe_if)),$$

where fHe_i is projective by Lemma B.5.1. The left equality follows from

$$\begin{aligned} f\operatorname{rad}(He_i) &= f\mathfrak{J}(H)e_i \\ &= f\mathfrak{J}(H)f(fe_if) \\ &= \mathfrak{J}(fHf)(fe_if) \\ &= \operatorname{rad}((fHf)(fe_if)) \\ &= \operatorname{rad}(fHe_i), \end{aligned}$$

where we uses [Lam91, Theorem 21.10] about the relationship of Jacobson radicals and idempotents and Nakayama's lemma. Therefore, the diagram induces the dotted isomorphism, proving that the fS with $S \in \mathcal{S}_r \cup \mathcal{S}_i$ are indeed precisely the simple modules in fHf.

We still need to show that $S \cong S'$ if and only of $fS \cong fS'$. The implication to the right is clear, and the implication to the left follows from the fact that $eS = eS' = 0$ by Lemma B.5.2 and that the restriction functor r_f from the category of all H-modules M with $eM = 0$ to the category of fHf-modules is fully faithful according to Lemma B.5.3 (i). This finishes the proof. □

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